Lecture 18: (Undirected) Graphs

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Definitions

\[ E(G) \subseteq V(G) \times V(G) \text{ where } E(G) \text{ is symmetric and irreflexive.} \]

Graph. A (simple, undirected) graph \( G \) consists of a non-empty set of vertices \( V(G) \) and a set of edges \( E(G) \). Each edge \( e \in E(G) \) is a two element subset of \( V(G) \), i.e., it is of the form \( \{u, v\} \) where \( u \neq v \).

- Vertices \( u \) and \( v \) are said to be end points of edge \( \{u, v\} \).
- Edge \( \{u, v\} \) is said to be incident to \( u \) and \( v \).
- Vertices \( u \) and \( v \) are said to adjacent if \( \{u, v\} \in E(G) \).
- The degree of vertex \( v \), \( \deg(v) \), is the number of edges incident on \( v \).

Example Graphs.

Complete graph \( K_n \)

\[
V(K_n) = \{1, 2, \ldots, n\}, \\
E(K_n) = \{i, j \mid 1 \leq i < j \leq n \text{ and } i \neq j\}
\]

Cycle Graph \( C_n \)

\[
V(C_n) = \{1, 2, \ldots, n\}, \\
E(C_n) = \{i, i+1 \mid 1 \leq i \leq n\} \cup \{n, 1\}
\]

Wheel Graph \( W_n \)

\[
V(W_n) = \{0, 1, \ldots, n\} \\
E(W_n) = \{0, 1, \ldots, n\} \cup \{i, i+1 \mid 1 \leq i < n\} \cup \{n, 0\}
\]

Problem 1. On average, who has more opposite-gender partners: men or women?

\[
\text{deg}(v) = \# \text{ opp. gender partners } v \text{ has had} \\
\text{Average # partners men} = \frac{\sum_{v \in M} \text{deg}(v)}{|M|} / \frac{\sum_{v \in W} \text{deg}(v)}{|W|}
\]

\[
\sum_{v \in M} \text{deg}(v) = \sum_{v \in W} \text{deg}(v) \\
\frac{\sum_{v \in M} \text{deg}(v)}{|M|} = \frac{\sum_{v \in W} \text{deg}(v)}{|W|}
\]

Average # partners for men = \frac{|M|}{1M1}

Average # partners for women = \frac{|W|}{1W1}
Problem 2. Alice and Bob are describing a party they both attended where every person at the party shook hands with exactly 5 other people. However, Alice and Bob disagree on how many people were there at the party. Alice claims there were 125 people, while Bob claims there were 173. Who among Alice and Bob is definitely wrong?

\[ \text{Attendees - Vertices} \quad \text{deg}(v) = 5 \]

\[ \text{Edges - Shook hands} \quad \text{Handshaking Lemma: } \sum_{v \in V(G)} \text{deg}(v) = 2 |E| \]

Bob: \[ \sum_{v \in V(G)} \text{deg}(v) = 173 \times 5 \quad \times \]

Alice: \[ \sum_{v \in V(G)} \text{deg}(v) = 126 \times 5 \]

Corollary: In any undirected graph, the number of odd vertices is even.

Isomorphism

Definition. An isomorphism between graphs \( G \) and \( H \) is a bijection \( f : V(G) \rightarrow V(H) \) such that

\[ \{u, v\} \in E(G) \iff \{f(u), f(v)\} \in E(H). \]

\( G \) and \( H \) are said to be isomorphic if there is (some) isomorphism between \( G \) and \( H \).

Question 1. Let \( G \) and \( H \) be isomorphic graphs. For each of the following statements decide if it is necessarily true. (a) \( V(G) = V(H) \) \( \times \) (b) \( |V(G)| = |V(H)| \) \( T \) (c) \( E(G) = E(H) \) \( F \) (d) \( |E(G)| = |E(H)| \) \( T \)