Lecture 18: (Undirected) Graphs

Date: October 11, 2019.

Definitions

$E(G) \subseteq V(G) \times V(G)$ where $E(G)$ is symmetric and irreflexive

Graph. A (simple, undirected) graph G consists of a non-empty set of vertices $V(G)$ and a set of edges $E(G)$. Each edge $e \in E(G)$ is a two element subset of $V(G)$, i.e., it is of the form $\{u, v\}$ where $u \neq v$.

- Vertices u and v are said to be **end points** of edge $\{u, v\}$.
- Edge $\{u, v\}$ is said to be **incident** to u and v.
- Vertices u and v are said to **adjacent** if $\{u, v\} \in E(G)$.
- The **degree** of vertex v, $\deg(v)$, is the number of edges incident on v.

Example Graphs.

Complete graph K_n

K_4

$V(K_n) = \{1, 2 \dots n\}$,
$E(K_n) = \{\{i, j\} \mid 1 \le i, j \le n \text{ AND } i \neq j\}$

Cycle Graph C_n

C_5

C_4

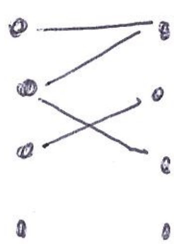
$V(C_5) = \{1, 2, \dots n\}$,
$E(C_5) = \{\{u, v\} \mid |u - v| = 1\}$
$\cup \{\{1, n\}\}$

Wheel Graph W_n

W_5

$V(W_n) = \{0, 1, \dots n\}$
$E(W_n) = \{\{0, i\} \mid i \in \{1 \dots n\}\}$
$\cup \{\{i, j\} \mid 1 \le i, j \le n \wedge |i - j| = 1\}$
$\cup \{\{1, n\}\}$

Problem 1. On average, who has more opposite-gender partners: men or women?



M W

$\deg(v) = \#$ opp-gender partners v has had

Average # partners men / women $= \dfrac{\sum_{v \in M} \deg(v)}{|M|} \Big/ \dfrac{\sum_{v \in W} \deg(v)}{|W|}$

$$\sum_{v \in M} \deg(v) = \sum_{v \in W} \deg(v)$$

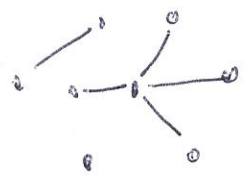
$$|M| \underbrace{\left(\frac{\sum_{v \in M} \deg(v)}{|M|} \right)}_{\text{Ave for men}} = |W| \underbrace{\left(\frac{\sum_{v \in W} \deg(v)}{|W|} \right)}_{\text{Ave for women}}$$

$$\frac{\text{Average \# partners for men}}{\text{Average \# partners for women}} = \frac{|W|}{|M|}$$

Problem 2. Alice and Bob are describing a party they both attended where every person at the party shook hands with exactly 5 other people. However, Alice and Bob disagree on how many people were there at the party. Alice claims there were 126 people, while Bob claims there were 173. Who among Alice and Bob is definitely wrong?

Attendees – Vertices $\quad \deg(v) = 5$

Edges – Shook hands $\quad$ Handshaking Lemma: $\sum_{v \in V(G)} \deg(v) = 2|E|$

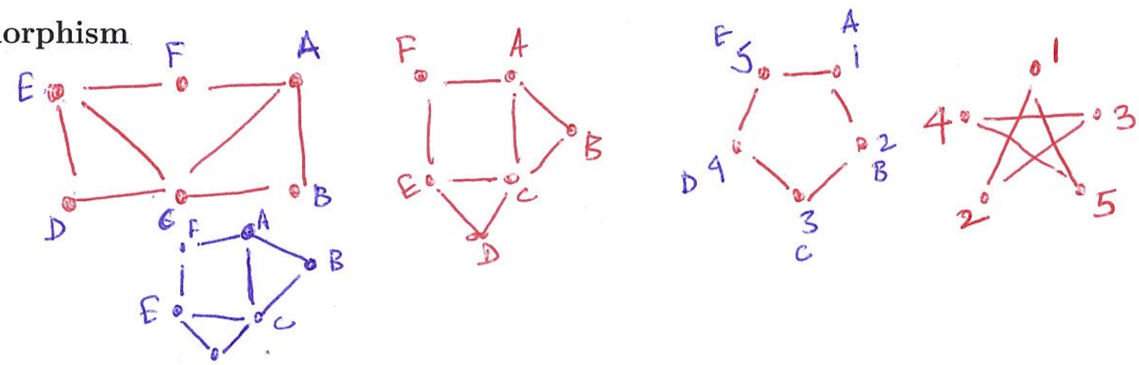


Bob: $\sum_{v \in V(G)} \deg(v) = \sum_{v \in V(G)} 5 = 173 \times 5$ ✗

Alice: $\sum_{v \in V(G)} \deg(v) = 126 \times 5$

Corollary: In any undirected graph, ~~the~~ the number of odd vertices is even.

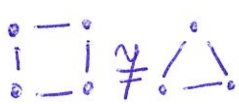
Isomorphism



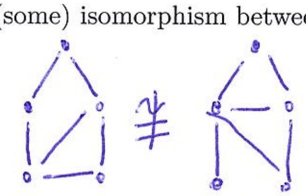
Definition. An **isomorphism** between graphs G and H is a bijection $f : V(G) \to V(H)$ such that

$$\{u, v\} \in E(G) \text{ IFF } \{f(u), f(v)\} \in E(H).$$

G and H are said to **isomorphic** if there is (some) isomorphism between G and H.



vertices are not equal.



There is no deg 1 vertex in the left graph.

Question 1. Let G and H be isomorphic graphs. For each of the following statements decide if it is necessarily true. (a) $V(G) = V(H)$ F (b) $|V(G)| = |V(H)|$ T (c) $E(G) = E(H)$ F (d) $|E(G)| = |E(H)|$ T