Lecture 10: More Induction

Date: September 20, 2019.

Induction: To prove \( \forall n \in \mathbb{N} \) such that \( n \geq b \), \( P(n) \)

- \( P(b) \) [Base Case]
- Prove for all \( n > b \), if \( P(b) \) AND \( P(b+1) \) AND \( \cdots \) AND \( P(n-1) \) then \( P(n) \) [Induction Step]

**Proposition 1.** For any \( n \geq 0 \), a \( 2^n \times 2^n \) checker board with a “middle square” removed can be tiled using L-shaped triominoes.

**Induction Predicate:**
\( P(n) \): A \( 2^n \times 2^n \) grid with middle removed can be tiled using triominoes.

**Base Case:** Need to prove \( P(0) \)
\( P(0) \) holds trivially.

**Ind Hyp:** Assume \( P(0), P(1), \ldots, P(n-1) \) hold.

**Ind Step:** Need to prove \( P(n) \).

\( Q(n) \): A \( 2^n \times 2^n \) grid with any square removed can be tiled using L-shape tiles.

**Base Case:** \( Q(0) \) holds trivially.

**Ind Hyp:** Assume \( Q(0), Q(1), \ldots, Q(n-1) \) hold.

**Ind Step:** To prove \( Q(n) \).

By \( Q(n) \), \( U \) by \( Q(n-1) \).

By Ind Hyp, \( R \) with corner removed can be tiled.

By induction principle \( \forall n \), \( Q(n) \)
That implies \( \forall n \), \( P(n) \)
F(0) = 0, F(1) = 1, F(2) = 1, F(3) = 2, F(4) = 3, F(5) = 5, F(6) = 8, F(7) = 13, ...

Fibonacci Numbers: Numbers obtained by the following recursive process: F(0) = 0, F(1) = 1, and
F(n) = F(n-1) + F(n-2) when n > 1.

Proposition 2. For any n ≥ 0, F(n) is even IFF F(n+3) is even.

Induction Predicate: P(n) := F(n) is even IFF F(n+3) is even.

Base Case: n = 0: F(0) = 0 is even IFF F(3) = 2 is even.

Ind. Hyp.: P(0), P(1), ..., P(n-1) hold.

Ind. Step: Prove F(n) is even IFF F(n+3) is even.

F(n) is even <=> F(n-1) + F(n) is even.

 <=> [F(n-1) is even IFF F(n-2) is even]

 <=> [F(n+1) is even IFF F(n+2) is even] (Ind. Hyp.)

 <=> F(n+1) + F(n+2) is even

 <=> F(n+3) is even

Theorem 3. Every integer greater than 1 is a product of primes.

(Weak) Induction: To prove \( \forall n \in \mathbb{N} \) such that \( n \geq b \), P(n)

- Prove P(b) [Base Case]

- Prove for all \( n > b \), if P(n-1) then P(n) [Induction Step]