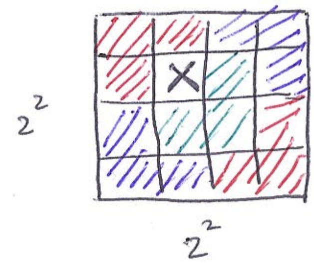
Lecture 10: More Induction

Date: September 20, 2019.

Induction: To prove $\forall n \in \mathbb{N}$ such that $n \geq b$, $P(n)$

- Prove $P(b)$ **[Base Case]**
- Prove for all $n > b$, if $P(b)$ AND $\cancel{P(1)}$ $P(b+1)$ AND $\cdots$ AND $P(n-1)$ then $P(n)$ **[Induction Step]**

Proposition 1. *For any $n \geq 0$, a $2^n \times 2^n$ checker board with a "middle square" removed can be tiled using L-shaped triominoes.*

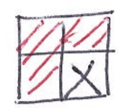


Induction Predicate.
$P(n)$: A $2^n \times 2^n$ grid with middle removed can be tiled using triominoes

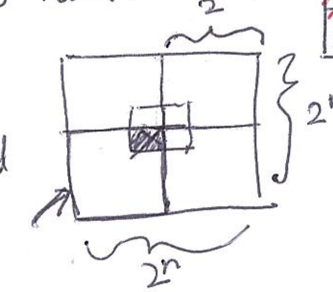
Base Case: Need to prove $P(0)$

$P(0)$ holds trivially

Ind Hyp: Assume $P(0), P(1), \ldots P(n-1)$ hold.



Ind Step: Need to prove $P(n)$

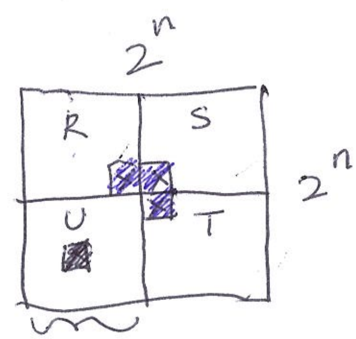


$Q(n)$: A $2^n \times 2^n$ grid with any square removed can be tiled using L-shape tiles.

Base Case: $Q(0)$ holds trivially.

Ind Hyp: Assume $Q(0), Q(1) \ldots Q(n-1)$ hold.

Ind Step: To prove $Q(n)$.



We can tile grid U by $Q(n-1)$

By Ind hyp, R with corner removed can be tiled

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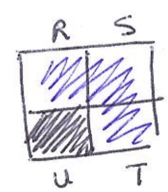

By induction principle $\forall n\ Q(n)$
That implies $\forall n\ P(n)$

$F(0)=0,\ F(1)=1,\ F(2)=1,\ F(3)=2,\ F(4)=3,\ F(5)=5,\ F(6)=8,\ F(7)=13, \ldots$

Fibonacci Numbers: Numbers obtained by the following recursive process: $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ when $n > 1$.

Proposition 2. *For any $n \geq 0$, $F(n)$ is even* IFF *$F(n+3)$ is even.*

Induction Predicate: $P(n) ::=$ $F(n)$ is even iff $F(n+3)$ is even.

Base Case: $n=0$: $F(0)=0$ is even iff $F(3)=2$ is even ✓

$n=1$: $F(1)=1$ odd iff $F(4)=3$ is odd ✓

Ind Hyp: $P(0), P(1) \ldots P(n-1)$ hold.

Ind Step: Prove $F(n)$ is even iff $F(n+3)$ is even.

$F(n)$ is even $\iff$ $F(n-2)+F(n-1)$ is even.

$\iff$ [$F(n-2)$ is even iff $F(n-1)$ is even]

$\iff$ [$F(n+1)$ is even iff $F(n+2)$ is even] (ind hyp)

$\iff$ $F(n+1)+F(n+2)$ is even

$\iff$ $F(n+3)$ is even

Theorem 3. *Every integer greater than 1 is a product of primes.*

(Weak) Induction: To prove $\forall n \in \mathbb{N}$ such that $n \geq b$, $P(n)$

- Prove $P(b)$ [**Base Case**]
- Prove for all $n > b$, if $P(n-1)$ then $P(n)$ [**Induction Step**]