Lecture 26: Sums, Products, and Bijections

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Sum Rule. If $A_1, A_2, \dots A_n$ are pairwise disjoint sets (i.e., $A_i \cap A_j = \emptyset$ for every $i \neq j$) then

$$|\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i|.$$

Problem 1. Suppose we roll a black die and a white die. In how many outcomes will the two dice show different values?

Complementary Counting. Suppose $A \subseteq S$. To find $|A|$, sometimes it is easier to find $|U|$ and $|U - A|$; then $|A| = |U| - |U - A|$.

Product Rule. If $A_1, A_2, \dots A_n$ are finite sets, then

$$|A_1 \times A_2 \times \cdots \times A_n| = \prod_{i=1}^{n} |A_i|.$$

Problem 2. How many binary strings of length n?

Problem 3. A restaurant menu has 5 appetizers, 6 entrees, 3 salads, and 7 desserts.

1. How many items are on the menu?

2. How many ways to choose a complete meal that consists of each course?

3. How many ways to order a meal if I may not choose some courses?

Problem 4. Suppose we roll a black die and a white die. In how many outcomes will the black die show a smaller value than the white die?

Correspondence Principle. For finite sets A and B

- If there is a surjection $F : A \to B$ then $|A| \geq |B|$.
- If there is a injection $f : A \to B$ then $|A| \leq |B|$.
- If there is a bijection $f : A \to B$ then $|A| = |B|$.

Proposition 1. *Number of subsets of a set A of size n is 2^n.*

Problem 5. A valid password is a sequence between 6 and 8 symbols. The first symbol must be a letter (upper or lower case) and the remaining symbols can either be a letter (upper or lower case) or a digit. How many passwords are there?

Generalized Product Rule. Let S be a set of length k sequences such that there are n_1 possibilities for the first entries, n_2 possibilities for the second entries for each first entry, $\ldots n_k$ possibilities for the kth entries for each sequence of first $k-1$ entries. Then $|S| = n_1 \cdot n_2 \cdot n_3 \cdots n_k$.

Problem 6. How many ways to order a deck with 52 cards?

Problem 7. A dollar bill is *defective* if some digit appears more than once in the 8-digit serial number. How many defective bills are there?