Lecture 26: Sums, Products, and Bijections

Date: November 6, 2019.

**Sum Rule.** If $A_1, A_2, \ldots, A_n$ are pairwise disjoint sets (i.e., $A_i \cap A_j = \emptyset$ for every $i \neq j$) then

$$|\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i|.$$  

**Problem 1.** Suppose we roll a black die and a white die. In how many outcomes will the two dice show different values?

**Complementary Counting.** Suppose $A \subseteq S$. To find $|A|$, sometimes it is easier to find $|U|$ and $|U - A|$; then $|A| = |U| - |U - A|$.

**Product Rule.** If $A_1, A_2, \ldots, A_n$ are finite sets, then

$$|A_1 \times A_2 \times \cdots \times A_n| = \prod_{i=1}^{n} |A_i|.$$  

**Problem 2.** How many binary strings of length $n$?

**Problem 3.** A restaurant menu has 5 appetizers, 6 entrees, 3 salads, and 7 desserts.

1. How many items are on the menu?

2. How many ways to choose a complete meal that consists of each course?

3. How many ways to order a meal if I may not choose some courses?

**Problem 4.** Suppose we roll a black die and a white die. In how many outcomes will the black die show a smaller value than the white die?
Correspondence Principle. For finite sets $A$ and $B$

- If there is a surjection $F : A \rightarrow B$ then $|A| \geq |B|$. 
- If there is a injection $f : A \rightarrow B$ then $|A| \leq |B|$. 
- If there is a bijection $f : A \rightarrow B$ then $|A| = |B|$. 

**Proposition 1.** Number of subsets of a set $A$ of size $n$ is $2^n$.

**Problem 5.** A valid password is a sequence between 6 and 8 symbols. The first symbol must be a letter (upper or lower case) and the remaining symbols can either be a letter (upper or lower case) or a digit. How many passwords are there?

**Generalized Product Rule.** Let $S$ be a set of length $k$ sequences such that there are $n_1$ possibilities for the first entries, $n_2$ possibilities for the second entries for each first entry, \ldots $n_k$ possibilities for the $k$th entries for each sequence of first $k-1$ entries. Then $|S| = n_1 \cdot n_2 \cdot \ldots \cdot n_k$.

**Problem 6.** How many ways to order a deck with 52 cards?

**Problem 7.** A dollar bill is defective if some digit appears more than once in the 8-digit serial number. How many defective bills are there?