Lecture 30: Pigeonhole Principle

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Subset Split Rule/Multinomial Coefficient. The expression

$$\binom{k_1 + k_2 + \cdots + k_m}{k_1, k_2, \ldots k_m} = \frac{(k_1 + k_2 + \cdots + k_m)!}{k_1! k_2! \cdots k_m!}$$

is the number of ways

- of forming m distinct subsets of sizes $k_1, k_2, \ldots k_m$ (respectively) out of a set of $(k_1 + k_2 + \cdots + k_m)$ elements;
- of the number of sequences formed from $l_1, l_2, \ldots l_m$, where the sequence has k_1 copies of l_1, k_2 copies of l_2, $\ldots k_m$ copies of l_m in the sequence.

Binomial Theorem. $(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$.

Problem 1. What is the coefficient of $be^3k^2o^2pr$ in the expansion of $(b + e + k + o + p + r)^{10}$?

Pigeonhole Principle. If $|A| > |B|$ then for every function $f : A \to B$, there exist distinct $a, b \in A$ such that $f(a) = f(b)$.

Problem 2. Let S be any n-element set of integers. There are $a, b \in S$ such that $n - 1 \mid (a - b)$.

Problem 3. A chess player trains for a championship by playing practice games over 77 days. She plays at least one game on any day, and plays a total of at most 132 games. Prove that no matter what her schedule of games looks like, there is a period of consecutive days in which she plays **exactly** 21 games.

Generalized Pigeonhole Principle. Let $B = \{b_1, b_2, \dots b_n\}$. Let $q_1, q_2, \dots q_n \in \mathbb{N}$ be such that $|A| > q_1 + q_2 + \cdots + q_n$. For any function $f : A \to B$, for some i, $|\{a \in A \mid f(a) = b_i\}| > q_i$.

- If $|A| > k|B|$ then for every function $f : A \to B$, there are $k+1$ distinct elements of A $a_1, a_2, \dots a_{k+1}$ such that for every i, j, $f(a_i) = f(a_j)$.

Problem 4.

1. How many cards should you pick from a standard deck of 52 cards to guarantee that at least 3 cards of the same suit are chosen?
2. How many cards should you pick from a standard deck of 52 cards to guarantee that at least 3 cards from the "Hearts" suit are picked?

Subsequence. For a sequence $a_1, a_2, \dots a_n$, a subsequence is a sequence of the form $a_{i_1}, a_{i_2}, \dots a_{i_k}$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

Theorem 1 (Erdös-Szekeres)**.** *Any sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length at least $n + 1$ that is either increasing or decreasing.*