Lecture 30: Pigeonhole Principle

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Subset Split Rule/Multinomial Coefficient. The expression
\[
\binom{k_1 + k_2 + \cdots + k_m}{k_1, k_2, \ldots, k_m} = \frac{(k_1 + k_2 + \cdots + k_m)!}{k_1!k_2!\cdots k_m!}
\]
is the number of ways
- of forming \(m\) distinct subsets of sizes \(k_1, k_2, \ldots, k_m\) (respectively) out of a set of \((k_1 + k_2 + \cdots + k_m)\) elements;
- of the number of sequences formed from \(l_1, l_2, \ldots, l_m\), where the sequence has \(k_1\) copies of \(l_1\), \(k_2\) copies of \(l_2\), \ldots \(k_m\) copies of \(l_m\) in the sequence.

Binomial Theorem. \((x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}\).

Problem 1. What is the coefficient of \(be^3k^2o^2pr\) in the expansion of \((b + e + k + o + p + r)^{10}\)?

Pigeonhole Principle. If \(|A| > |B|\) then for every function \(f : A \rightarrow B\), there exist distinct \(a, b \in A\) such that \(f(a) = f(b)\).

Problem 2. Let \(S\) be any \(n\)-element set of integers. There are \(a, b \in S\) such that \(n - 1 \mid (a - b)\).
**Problem 3.** A chess player trains for a championship by playing practice games over 77 days. She plays at least one game on any day, and plays a total of at most 132 games. Prove that no matter what her schedule of games looks like, there is a period of consecutive days in which she plays exactly 21 games.

**Generalized Pigeonhole Principle.** Let $B = \{b_1, b_2, \ldots, b_n\}$. Let $q_1, q_2, \ldots, q_n \in \mathbb{N}$ be such that $|A| > q_1 + q_2 + \cdots + q_n$. For any function $f : A \rightarrow B$, for some $i$, $|\{a \in A \mid f(a) = b_i\}| > q_i$.

- If $|A| > k|B|$ then for every function $f : A \rightarrow B$, there are $k + 1$ distinct elements of $A \, a_1, a_2, \ldots, a_{k+1}$ such that for every $i, j$, $f(a_i) = f(a_j)$.

**Problem 4.**
1. How many cards should you pick from a standard deck of 52 cards to guarantee that at least 3 cards of the same suit are chosen?
2. How many cards should you pick from a standard deck of 52 cards to guarantee that at least 3 cards from the “Hearts” suit are picked?

**Subsequence.** For a sequence $a_1, a_2, \ldots, a_n$, a subsequence is a sequence of the form $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

**Theorem 1** (Erdős-Szekeres). Any sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length at least $n + 1$ that is either increasing or decreasing.