# Lecture 35: Infinite Sets

Date: December 4, 2019.

**Comparing the sizes of Infinite sets.**

**Proposition 1.** *The following statements hold for finite sets $A$ and $B$.*

1. *If there is a surjective function $f : A \to B$ then $|A| \geq |B|$.*
2. *If there is a injective function $f : A \to B$ then $|A| \leq |B|$.*
3. *If there is a bijective function $f : A \to B$ then $|A| = |B|$.*

**Cantor's Definition.** For infinite sets $A$ and $B$, we will say $|A| \leq |B|$ if there is a injective function $f : A \to B$.

- We will say $|A| = |B|$ if there is a bijective function $f : A \to B$.

**Proposition 2.** *For any (non-empty) sets $A, B$, and $C$ the following properties hold.*

1. *If there are injective functions $f : A \to B$, and $g : B \to C$ then there is an injective function $h : A \to C$.*
2. *If there is an injective function $f : A \to B$ then there is a surjective function $g : B \to A$.*

**Theorem 3** (Cantor-Schröder-Bernstein)**.** *For any sets $A$ and $B$, if there are injective functions $f : A \to B$ and $g : B \to A$ then there is a bijective function $h : A \to B$.*

**Problem 1.** Let $\mathbb{E} = \{n \in \mathbb{N} \mid n \equiv 0 \pmod 2\}$. Show that $|\mathbb{E}| \leq |\mathbb{N}|$ and $|\mathbb{N}| \leq |\mathbb{E}|$.

**Problem 2.** Show that $|\mathbb{Z}| \leq |\mathbb{N}|$.

**Problem 3.** Show that $|\mathbb{N} \times \mathbb{N}| \leq |\mathbb{N}|$.

**Problem 4.** Show that $|\mathbb{Q}| \leq |\mathbb{N}|$.

**Countable Sets.** A (finite or infinite) set $A$ is said to be **countable** if there is an injective function $f : A \to \mathbb{N}$. In other words, if $|A| \leq |\mathbb{N}|$.

**Theorem 4** (Cantor)**.** *For any set $A$, $\mathsf{pow}(A) \not\leq A$*