# Lecture 8: Finite Cardinality and Induction

Date: September 18, 2019.

**Sequences on $A$:** Ordered list of elements from $A$.

- Length two sequences $(a_1, a_2)$, i.e., pairs, i.e., element of $A \times A$
- Length $n$ sequences $(a_1, a_2, \dots a_n) \in A \times A \times \cdots A$

**Bijective Functions:**

- $f : A \to B$ is surjective/onto if $\mathsf{range}(f) = f(A) = B = \mathsf{codomain}(f)$.
- $f : A \to B$ is injective/1-to-1 if *distinct* elements get mapped to *distinct* elements.
- A function is bijective if it is injective/1-to-1 and surjective/onto.

**Cardinality (of finite sets):** $|X|$ = number of elements in $X$.

**Example 1.** $|\emptyset| =$ $|\{0, 1, 2, 3\}| =$ $|\{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}| =$ $|\{0, 1, 1, 2, 2\}| =$
$|\{0, 1, 2\} \times \{a, b, c\}| =$ |sequences of length $n$ over $\{0, 1, 2\}$| =

**Proposition 1.** *The following statements hold for finite sets $A$ and $B$.*

1. *If there is a surjective function $f : A \to B$ then $|A| \geq |B|$.*
2. *If there is a injective function $f : A \to B$ then $|A| \leq |B|$.*
3. *If there is a bijective function $f : A \to B$ then $|A| = |B|$.*

**Proposition 2.** *For a set $A$ such that $|A| = n$, $|\mathsf{pow}(A)| = 2^n$.*

**Induction:** To prove $\forall n \in \mathbb{N}\ P(n)$

- Prove $P(0)$ **[Base Case]**
- Prove for all $n > 0$, if $P(0)$ $\mathsf{AND}$ $P(1)$ $\mathsf{AND}$ $\cdots \mathsf{AND}$ $P(n-1)$ then $P(n)$ **[Induction Step]**

**Proposition 3.** *Prove for all $n \in \mathbb{N}$*

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

**Proposition 4.** *Prove that for all $n \in \mathbb{N}$, $\sum_{i=0}^{n} i2^i = (n-1)2^{n+1} + 2$.*

**Problem 1.** All horses have the same color.

Proof by induction. Predicate $P(n)$: Any set of $n$-horses has the same color. To prove: $\forall n \in \mathbb{N}$ with $n \geq 1$, $P(n)$

Base Case: $P(1)$. In any set containing only one horse, all horses (namely the only one) have the same color.

Induction Hypothesis: Assume that $P(1)$, $P(2)$, $\ldots P(n-1)$ are true.

Induction Step: Consider an arbitrary set $H$ of $n+1$ horses.

Let $H = \{h_1, h_2, \ldots h_n\}$

Consider $H_1 = \{h_1, h_2, \ldots h_{n-1}\}$ and $H_2 = \{h_2, \ldots h_n\}$

Since $P(n-1)$ holds, all horses in $H_1$ have the same color. Also all horses in $H_2$ have the same color.

So color$(h_1)$ = color$(h_2)$ = color$(h_3)$ $= \cdots =$ color$(h_n)$. Hence all horses in $H$ have the same color.