Lecture 19: Subgraphs and Connectivity

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Isomorphism

Definition. An **isomorphism** between graphs G and H is a bijection $f : V(G) \to V(H)$ such that

$$\{u, v\} \in E(G) \text{ IFF } \{f(u), f(v)\} \in E(H).$$

G and H are said to **isomorphic** if there is (some) isomorphism between G and H.

Degree Sequence of a graph G is a listing of the degrees of the vertices of G in ascending order.

Proposition 1. *If G and H are isomorphic then they have the same degree sequence.*

Subgraphs. G is a subgraph of H iff $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$.

Proposition 2. *Let G and H be isomorphic graphs. If S is a subgraph of G then there is a graph T such that T is a subgraph of H such that S and T are isomorphic.*

Walks, Paths, and Cycles

Walk in graph G is an alternating sequence of vertices and edges that begins with a vertex, ends with a vertex, and for any edge $e = \{u, v\}$ in the walk, one of its endpoints is just before e in the sequence and the other endpoint is just after e.

$$\text{Walk is of the form } v_0\{v_0, v_1\}v_1\{v_1, v_2\}v_2 \cdots \{v_{k-1}, v_k\}v_k.$$

The **length** of a walk is the number of edges in it.

Path is a walk such that all vertices appearing in it are distinct.

Closed Walk is a walk that begins and ends in the same vertex.

Cycle is a closed walk of length > 2 such that all vertices are distinct except the first and the last.

Connectivity. Vertices u and v are **connected** in graph G if there is a path that starts in u and ends in v. We denote this by $\mathsf{conn}(u, v)$. A graph G is **connected** if every pair of vertices are connected.

Proposition 3. conn *is an equivalence relation.*

Connected Components. Equivalence classes of conn are the **connected components** of a graph G.

Special Walks and Tours

Eulerian Tour of G is a closed walk that includes every edge exactly once.

Theorem 4. *A connected graph has an Eulerian tour if and only if every vertex has an even degree.*

Hamiltonian Cycle of G is a cycle that visits every vertex in G exactly once.