Lecture 8: Functions, Binary Relations, and Cardinality

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**Functions:** A function $f : A \to B$ assigns to an element of one set the **domain** (in this case $A$), an element from another set the **codomain** (in this case $B$).

**Example 1.** inc : $\{0, 1, 2\} \to \{0, 1, 2\}$ where inc(0) = 1, inc(1) = 2, and inc(2) = 0.

dbl : $\mathbb{N} \to \mathbb{N}$ wheredbl(n) = 2n.

twoinc : $\mathbb{Z} \to \mathbb{Z}$ where twoinc(x) = x + 2.

sq : $\mathbb{R} \to \mathbb{R}$ where sq(x) = $x^2$.

**Evaluation on Sets:** Given a function $f : A \to B$ and $S \subseteq A$, $f(S) = \{f(n) \mid n \in S\} \subseteq B$.

**Example 2.** inc($\{0, 1\}$) =
dbl($\mathbb{N}$) =

**Range:** The range of $f : A \to B$ is the set $f(A)$.

**Surjective/Onto:** $f : A \to B$ is surjective/onto if range($f$) = $f(A) = B = \text{codomain}(f)$, i.e.,

$$\forall y \in B \exists x \in A (f(x) = y)$$

**Question 1.** Which of the following functions is surjective? (a) inc, (b) dbl, (c) twoinc, (d) sq

**Injective/1-to-1:** $f : A \to B$ is injective/1-to-1 if distinct elements get mapped to distinct elements, i.e.,

$$\forall x \in A \forall y \in A ((x \neq y) \implies (f(x) \neq f(y)))$$

**Question 2.** Which of the following functions is injective? (a) inc, (b) dbl, (c) twoinc, (d) sq

**Composition:** For functions $f : A \to B$ and $g : B \to C$, the composition $g \circ f$ is the function $A \to C$ defined as $(g \circ f)(x) = g(f(x))$, for all $x \in A$.

**Problem 1.** If $f : A \to B$ and $g : B \to C$ are injective then $g \circ f$ is injective.

**Proposition 1.** If $f : A \to B$, $g : B \to C$, and $g$ is surjective then $g \circ f$ is surjective.

**Bijective:** A function that is injective/1-to-1 and surjective/onto.
**Binary Relation:** $R \subseteq A \times B$, where $A$ is the domain, and $B$ is the codomain.

**Notation:** $(a, b) \in R$ or $aRb$ or $R(a, b)$

**Example 3.** For any function $f : A \to B$, $\text{graph}(f) = \{(x, f(x)) \mid x \in A\}$.

“less than” is a binary relation from $\mathbb{R}$ to $\mathbb{R}$.

Consider the relation $\text{teaches} \subseteq \text{Instructor} \times \text{Courses}$. It may have tuples of the form $(\text{viswanathan}, \text{CS173BL1}), (\text{viswanathan}, \text{CS173AL1}), (\text{forbes}, \text{CS473}), (\text{chekuri}, \text{CS473})$...

**Cardinality (of finite sets):** $|X| = \text{number of elements in } X$.

**Example 4.** $|\emptyset| = |\{0, 1, 2, 3\}| = |\{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}| = |\{0, 1, 1, 2, 2\}| = |\{0, 1, 2\} \times \{a, b, c\}| = $

**Proposition 2.** The following statements hold for finite sets $A$ and $B$.

1. If there is a surjective function $f : A \to B$ then $|A| \geq |B|$.
2. If there is an injective function $f : A \to B$ then $|A| \leq |B|$.
3. If there is a bijective function $f : A \to B$ then $|A| = |B|$.

**Proposition 3.** For a set $A$ such that $|A| = n$, $|\text{pow}(A)| = 2^n$. 

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