Lecture 8: Functions, Binary Relations, and Cardinality

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Functions: A function $f : A \to B$ assigns to an element of one set the **domain** (in this case A), an element from another set the **codomain** (in this case B).

Example 1. $\mathsf{inc} : \{0, 1, 2\} \to \{0, 1, 2\}$ where $\mathsf{inc}(0) = 1$, $\mathsf{inc}(1) = 2$, and $\mathsf{inc}(2) = 0$.

$\mathsf{dbl} : \mathbb{N} \to \mathbb{N}$ where $\mathsf{dbl}(n) = 2n$.

$\mathsf{twoinc} : \mathbb{Z} \to \mathbb{Z}$ where $\mathsf{twoinc}(x) = x + 2$.

$\mathsf{sq} : \mathbb{R} \to \mathbb{R}$ where $\mathsf{sq}(x) = x^2$.

Evaluation on Sets: Given a function $f : A \to B$ and $S \subset A$, $f(S) = \{f(n) \mid n \in S\} \subseteq B$.

Example 2. $\mathsf{inc}(\{0, 1\}) =$
$\mathsf{dbl}(\mathbb{N}) =$

Range: The range of $f : A \to B$ is the set $f(A)$.

Surjective/Onto: $f : A \to B$ is surjective/onto if $\mathsf{range}(f) = f(A) = B = \mathsf{codomain}(f)$, i.e.,

$$\forall y \in B\ \exists x \in A (f(x) = y)$$

Question 1. Which of the following functions is surjective? (a) inc, (b) dbl, (c) twoinc, (d) sq

Injective/1-to-1: $f : A \to B$ is injective/1-to-1 if *distinct* elements get mapped to *distinct* elements, i.e.,

$$\forall x \in A\ \forall y \in A ((x \neq y) \text{ IMPLIES } (f(x) \neq f(y)))$$

Question 2. Which of the following functions is injective? (a) inc, (b) dbl, (c) twoinc, (d) sq

Composition: For functions $f : A \to B$ and $g : B \to C$, the composition $g \circ f$ is the function $A \to C$ defined as $(g \circ f)(x) = g(f(x))$, for all $x \in A$.

Problem 1. If $f : A \to B$ and $g : B \to C$ are injective then $g \circ f$ is injective.

Proposition 1. *If $f : A \to B$, $g : B \to C$, and g is surjective then $g \circ f$ is surjective.*

Bijective: A function that is injective/1-to-1 **and** surjective/onto.

Binary Relation: $R \subseteq A \times B$, where A is the *domain*, and B is the *codomain*.

Notation: $(a, b) \in R$ or aRb or $R(a, b)$

Example 3. For any function $f : A \to B$, $\mathsf{graph}(f) = \{(x, f(x)) \mid x \in A\}$.

"less than" is a binary relation from $\mathbb{R}$ to $\mathbb{R}$.

Consider the relation $\mathsf{teaches} \subseteq \mathsf{Instructor} \times \mathsf{Courses}$. It may have tuples of the form
`...(viswanathan, CS173BL1), (viswanathan, CS173AL1), ...(forbes, CS473), (chekuri, CS473) ...`

Cardinality (of finite sets): $|X|$ = number of elements in X.

Example 4. $|\emptyset| =$ $|\{0, 1, 2, 3\}| =$ $|\{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}| =$ $|\{0, 1, 1, 2, 2\}| =$
$|\{0, 1, 2\} \times \{a, b, c\}| =$

Proposition 2. *The following statements hold for finite sets A and B.*

1. *If there is a surjective function $f : A \to B$ then $|A| \geq |B|$.*
2. *If there is a injective function $f : A \to B$ then $|A| \leq |B|$.*
3. *If there is a bijective function $f : A \to B$ then $|A| = |B|$.*

Proposition 3. *For a set A such that $|A| = n$, $|\mathsf{pow}(A)| = 2^n$.*