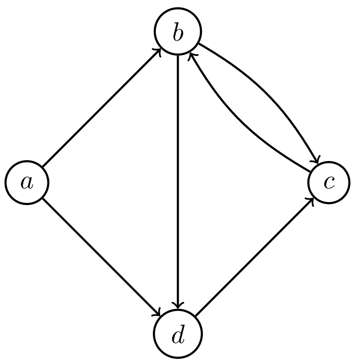
Lecture 16: Directed Graphs

Date: October 7, 2019.

Directed Graphs. G consists of nonempty set $V(G)$ of **vertices** (or **nodes**) and a set $E(G)$ of **edges**. Here $E(G) \subseteq V(G) \times V(G)$. An edge (u, v) has **source/tail** u and **target/head** v. A directed graph $G = (V(G), E(G))$ is also called a **digraph**.



Degrees. For a vertex $v \in V(G)$ of digraph G

$$\mathsf{indeg}(v) = |\{(u, v) \mid u \in V(G)\}|$$
$$\mathsf{outdeg}(v) = |\{(v, u) \mid u \in V(G)\}|$$

Proposition 1. *For any graph G, $\sum_{v \in V(G)} \mathsf{indeg}(v) = \sum_{v \in V(G)} \mathsf{outdeg}(v)$.*

Walks. A walk is an alternating sequence of vertices and edges that begins with a vertex, ends with a vertex, and such that for every edge (u, v) in the walk, u is the element just before the edge, and v is the element just after the edge in the sequence. So it is of the form

$$v_0(v_0, v_1)v_1(v_1, v_2) \cdots (v_{k-1}, v_k)v_k.$$

The walk is said to **start** in v_0 and end in v_k, and is of **length** k.

Simplification. A walk is completely determined by just the (sub-)sequence of vertices or the (sub-)sequence of edges. So we will just use that when convenient.

Paths. Is a walk, where each vertex in the sequence is distinct.

Closed Walk. Is a walk that starts and ends in the same vertex.

Cycle. Is a closed walk of length > 0 where all vertices except the first and last vertex are distinct.

Combining walks. If a walk $\mathbf{f}$ ends in vertex v and a walk $\mathbf{g}$ starts at the same vertex v, then they can be *merged* to get a longer walk. We will denote the merged walk by $\mathbf{f}\,\widehat{}\,\mathbf{g}$. Sometimes to emphasize the vertex where the walks merge, we will denote this by $\mathbf{f}\,\widehat{v}\,\mathbf{g}$.

Note, that $|\mathbf{f}\,\widehat{}\,\mathbf{g}| = |\mathbf{f}| + |\mathbf{g}|$.

Theorem 2. *A shortest walk between two vertices is a path.*

Distance. $\mathsf{dist}(u, v)$ is length of a shortest path from u to v.

Proposition 3. *For any graph G and vertices $u, v, w \in V(G)$, $\mathsf{dist}(u, w) \leq \mathsf{dist}(u, v) + \mathsf{dist}(v, w)$.*

Adjacency Matrix. A graph G with $V(G) = \{v_0, v_1, \ldots v_{n-1}\}$ can be represented by a matrix A_G where$(A_G)_{ij} = 1$ if $(v_i, v_j) \in E(G)$ and is 0 otherwise.

Length k-walk counting matrix. For graph G with vertices $\{v_0, v_1, \ldots v_{n-1}\}$, a length k walk counting matrix is a $n \times n$ matrix C such that C_{ij} = number of length k walks from v_i to v_j.

Theorem 4. *If C is a length k walk counting matrix, and D is a length m walk counting matrix, then CD is a length $k + m$ walk counting matrix.*