Lecture 16: Directed Graphs

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Directed Graphs. $G$ consists of nonempty set $V(G)$ of vertices (or nodes) and a set $E(G)$ of edges. Here $E(G) \subseteq V(G) \times V(G)$. An edge $(u, v)$ has source/tail $u$ and target/head $v$. A directed graph $G = (V(G), E(G))$ is also called a digraph.

Degrees. For a vertex $v \in V(G)$ of digraph $G$

\[
\text{indeg}(v) = |\{(u, v) \mid u \in V(G)\}|
\]

\[
\text{outdeg}(v) = |\{(v, u) \mid u \in V(G)\}|
\]

Proposition 1. For any graph $G$, $\sum_{v \in V(G)} \text{indeg}(v) = \sum_{v \in V(G)} \text{outdeg}(v)$.

Walks. A walk is an alternating sequence of vertices and edges that begins with a vertex, ends with a vertex, and such that for every edge $(u, v)$ in the walk, $u$ is the element just before the edge, and $v$ is the element just after the edge in the sequence. So it is of the form

$v_0(v_0, v_1)v_1(v_1, v_2)\cdots(v_{k-1}, v_k)v_k$.

The walk is said to start in $v_0$ and end in $v_k$, and is of length $k$.

Simplification. A walk is completely determined by just the (sub-)sequence of vertices or the (sub-)sequence of edges. So we will just use that when convenient.

Paths. Is a walk, where each vertex in the sequence is distinct.

Closed Walk. Is a walk that starts and ends in the same vertex.

Cycle. Is a closed walk of length $> 0$ where all vertices except the first and last vertex are distinct.

Combining walks. If a walk $f$ ends in vertex $v$ and a walk $g$ starts at the same vertex $v$, then they can be merged to get a longer walk. We will denote the merged walk by $f \hat{\triangle} g$. Sometimes to emphasize the vertex where the walks merge, we will denote this by $f \hat{\triangle} v g$.

Note, that $|f \hat{\triangle} g| = |f| + |g|$.
**Theorem 2.** A shortest walk between two vertices is a path.

**Distance.** \( \text{dist}(u, v) \) is length of a shortest path from \( u \) to \( v \).

**Proposition 3.** For any graph \( G \) and vertices \( u, v, w \in V(G) \), \( \text{dist}(u, w) \leq \text{dist}(u, v) + \text{dist}(v, w) \).

**Adjacency Matrix.** A graph \( G \) with \( V(G) = \{v_0, v_1, \ldots, v_{n-1}\} \) can be represented by a matrix \( A_G \) where \( (A_G)_{ij} = 1 \) if \((v_i, v_j) \in E(G)\) and is 0 otherwise.

**Length \( k \)-walk counting matrix.** For graph \( G \) with vertices \( \{v_0, v_1, \ldots, v_{n-1}\} \), a length \( k \) walk counting matrix is a \( n \times n \) matrix \( C \) such that \( C_{ij} = \) number of length \( k \) walks from \( v_i \) to \( v_j \).

**Theorem 4.** If \( C \) is a length \( k \) walk counting matrix, and \( D \) is a length \( m \) walk counting matrix, then \( CD \) is a length \( k + m \) walk counting matrix.