Lecture 35: Diagonalization

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Cantor's Definition. For infinite sets A and B, we will say $|A| \leq |B|$ if there is a injective function $f : A \to B$.

- If there is a surjective function $f : A \to B$ then $|A| \geq |B|$.
- We will say $|A| = |B|$ if there is a bijective function $f : A \to B$.

Countable Sets. A (finite or infinite) set A is said to be **countable** if there is an injective function $f : A \to \mathbb{N}$. In other words, if $|A| \leq |\mathbb{N}|$.

Proposition 1. *The sets $\mathbb{Z}$ and $\mathbb{N} \times \mathbb{N}$ are countable.*

Theorem 2 (Cantor)**.** *For any set A, $\mathsf{pow}(A) \not\leq A$*

Corollary 3. $\mathsf{pow}(\mathbb{N})$ *is not countable.*

Computational Problems and Programs

Proposition 4. *The number of programs is countable.*

Computational Problem. Each problem is a function that demands a certain answer be computed in response to an input.

Decision Problems. Problems that demand a Boolean answer in response to an input. Since every input is a binary string, decision problems are functions of type $\{0,1\}^* \to \{0,1\}$.

Proposition 5. *The number of decision problems is uncountable.*