Lecture 17: Directed Acyclic Graphs (DAGs) and Partial Orders

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Recap

- A digraph G consists of nonempty set $V(G)$ of vertices (or nodes) and a set $E(G)$ of edges.
- A walk is an alternating sequence of vertices and edges that begins with a vertex, ends with a vertex, and such that for every edge (u, v) in the walk, u is the element just before the edge, and v is the element just after the edge in the sequence.
- A paths is a walk, where each vertex in the sequence is distinct.
- A closed walk is a walk that starts and ends in the same vertex.
- A cycle is a closed walk of length > 0 where all vertices except the first and last vertex are distinct.
- A graph G with $V(G) = \{v_0, v_1, \ldots v_{n-1}\}$ can be represented by a matrix A_G where$(A_G)_{ij} = 1$ if $(v_i, v_j) \in E(G)$ and is 0 otherwise.

Walk Relations. For a digraph $G = (V(G), E(G))$, we define a couple of binary relations on $V(G)$ — a **walk relation** G^*, and a **positive walk relation** G^+. These are defined as follows.

$$(u, v) \in G^* \text{ IFF there is a walk from } u \text{ to } v$$
$$(u, v) \in G^+ \text{ IFF there is a walk of length} > 0 \text{ from } u \text{ to } v$$

Directed Acyclic Graphs (DAGs) Digraphs with no cycles.

Proposition 1. *Every DAG has a vertex v with* $\mathsf{indeg}(v) = 0$.

Topological Sort of a digraph G is a list of all vertices such that each if $(u, v) \in E(G)$ then u appears before v in the list.

Theorem 2. *Every DAG has a topological sort.*

Recap about Relations

- $R \subseteq A \times A$ is *reflexive* iff for all $a \in A$, $(a, a) \in R$.
- $R \subseteq A \times A$ is *symmetric* iff for all $a, b \in A$, $(a, b) \in R$ IMPLIES $(b, a) \in R$.
- $R \subseteq A \times A$ is *transitive* iff for all $a, b, c \in A$, $((a, b) \in R$ AND $(b, c) \in R)$ IMPLIES $(a, c) \in R$.

Proposition 3. *For a digraph G, the relations G^+ and G^* are transitive.*

Irreflexive. $R \subseteq A \times A$ is irreflexive iff for every $a \in A$, $(a, a) \notin R$.

Example: Which of the following relations on $\mathbb{N}$ is irreflexive? (a) $R = \emptyset$ (b) $R = \{(0, 0)\}$ (c) $R = \mathbb{N} \times \mathbb{N}$

Proposition 4. *If G is a DAG then G^+ is irreflexive.*

Strict Partial Orders. A relation $\prec A \times A$ that is transitive and irreflexive.

Examples: Standard ordering on natural numbers.
(strict) Subset ordering on sets.

Theorem 5. *A relation R is a strict partial order iff R is the positive walk relation for from DAG.*

Asymmetric. $R \subseteq A \times A$ is *asymmetric* iff for every $a, b \in A$, $(a, b) \in R$ IMPLIES $(b, a) \notin R$.

Proposition 6. *If R is a strict partial order then it is asymmetric.*

Antisymmetric. $R \subseteq A \times A$ is *antisymmetric* iff for all a, b such that $a \neq b$, $(a, b) \in R$ IMPLIES$(b, a) \notin R$.

(Weak) Partial Order. $\preceq A \times A$ is a partial order on A iff it is reflexive, transitive, and antisymmetric.

Theorem 7. *R is a partial order iff it is the walk relation of a DAG.*