Recap

- A digraph $G$ consists of nonempty set $V(G)$ of vertices (or nodes) and a set $E(G)$ of edges.
- A walk is an alternating sequence of vertices and edges that begins with a vertex, ends with a vertex, and such that for every edge $(u, v)$ in the walk, $u$ is the element just before the edge, and $v$ is the element just after the edge in the sequence.
- A paths is a walk, where each vertex in the sequence is distinct.
- A closed walk is a walk that starts and ends in the same vertex.
- A cycle is a closed walk of length $> 0$ where all vertices except the first and last vertex are distinct.
- A graph $G$ with $V(G) = \{v_0, v_1, \ldots v_{n-1}\}$ can be represented by a matrix $A_G$ where $(A_G)_{ij} = 1$ if $(v_i, v_j) \in E(G)$ and is 0 otherwise.

Walk Relations. For a digraph $G = (V(G), E(G))$, we define a couple of binary relations on $V(G) —$ a walk relation $G^*$, and a positive walk relation $G^+$. These are defined as follows.

$$(u, v) \in G^* \text{ IFF there is a walk from } u \text{ to } v$$

$$(u, v) \in G^+ \text{ IFF there is a walk of length } > 0 \text{ from } u \text{ to } v$$

Directed Acyclic Graphs (DAGs) Digraphs with no cycles.

Proposition 1. Every DAG has a vertex $v$ with $\text{indeg}(v) = 0$.

Topological Sort of a digraph $G$ is a list of all vertices such that each if $(u, v) \in E(G)$ then $u$ appears before $v$ in the list.

Theorem 2. Every DAG has a topological sort.
Recap about Relations

• $R \subseteq A \times A$ is reflexive iff for all $a \in A$, $(a,a) \in R$.
• $R \subseteq A \times A$ is symmetric iff for all $a, b \in A$, $(a,b) \in R$ IMPLIES $(b,a) \in R$.
• $R \subseteq A \times A$ is transitive iff for all $a, b, c \in A$, $(a,b) \in R$ AND $(b,c) \in R$ IMPLIES $(a,c) \in R$.

Proposition 3. For a digraph $G$, the relations $G^+$ and $G^*$ are transitive.

Irreflexive. $R \subseteq A \times A$ is irreflexive iff for every $a \in A$, $(a,a) \not\in R$.

Example: Which of the following relations on $\mathbb{N}$ is irreflexive? (a) $R = \emptyset$ (b) $R = \{(0,0)\}$ (c) $R = \mathbb{N} \times \mathbb{N}$

Proposition 4. If $G$ is a DAG then $G^+$ is irreflexive.

Strict Partial Orders. A relation $\prec A \times A$ that is transitive and irreflexive.

Examples: Standard ordering on natural numbers.
(strict) Subset ordering on sets.

Theorem 5. A relation $R$ is a strict partial order iff $R$ is the positive walk relation for from DAG.

Asymmetric. $R \subseteq A \times A$ is asymmetric iff for every $a, b \in A$, $(a,b) \in R$ IMPLIES $(b,a) \not\in R$.

Proposition 6. If $R$ is a strict partial order then it is asymmetric.

Antisymmetric. $R \subseteq A \times A$ is antisymmetric iff for all $a, b$ such that $a \neq b$, $(a,b) \in R$ IMPLIES $(b,a) \not\in R$.

(Weak) Partial Order. $\preceq A \times A$ is a partial order on $A$ iff it is reflexive, transitive, and antisymmetric.

Theorem 7. $R$ is a partial order iff it is the walk relation of a DAG.