CS 173 Discussion 7: Recursive Definitions, Structural Induction, and Graphs

Date: October 10/11, 2019.

Problem 1. Recall the following recursive definitions of strings and concatenation.

[Base]	$\lambda \in A^*$	[Base]	$\lambda \cdot t = t$
[Constructor]	if $a \in A,\ s \in A^*,$ then $\langle a, s\rangle \in A^*$	[Constructor]	$\langle a, s\rangle \cdot t = \langle a, s \cdot t\rangle$

Recall also that we proved in class that for any string s, $s \cdot \lambda = s$.

1. Let $\mathsf{rev}(s)$ denote the *reversal* of string s (i.e., s written backwards). Give a recursive definition of $\mathsf{rev}(s)$.
2. Prove that $\mathsf{rev}(s \cdot t) = \mathsf{rev}(t) \cdot \mathsf{rev}(s)$. You may assume (or prove it as an exercise at home) that concatenation is *associative*, i.e., for any strings r, s, t, $(r \cdot s) \cdot t = r \cdot (s \cdot t)$.

Problem 2 (de Bruijn sequences and Graphs)**.** Consider the set of length 3 strings over the binary alphabet $\{0, 1\}$. A *de Bruijn sequence* is binary string where every length 3 binary string appears at least once. For example, we could concatenate each length 3 string in the usual order (000, 001, and so on) to get a string of length $3 \times 8 = 24$ of the form $000001010\cdots$.

(a) Find a string of length 10 that is a de Bruijn sequence.

de Bruijn sequences can be found by constructing walks in a special graph called a *de Bruijn graph*. Such a graph D is defined as follows. The set of vertices $V(D) = \{ij \mid i, j \in \{0, 1\}\}$, and $E(D) = \{(ij, jk) \mid i, j, k \in \{0, 1\}\}$.

(b) Draw the de Bruijn graph.

(c) Explain how any walk that includes every edge in the de Bruijn graph determines a de Bruijn sequence.

(d) Explain how every de Bruijn sequence corresponds to a walk that visits every edge in the graph at least once.

(e) Use the above observations to argue that 10 is the minimum length for a de Bruijn sequence.

de Bruijn sequences and graphs as defined here have been specialized to the case of the binary alphabet (as opposed to a general alphabet) and length 3 strings (as opposed to a general length). See the wikipedia page (`https://en.wikipedia.org/wiki/De_Bruijn_sequence`) for more information on such graphs and sequences.