CS 173, Fall 2015
Examlet 8, Part B

NETID:

FIRST: **LAST:**

Discussion: **Thursday** **2** **3** **4** **5** **Friday** **9** **10** **11** **12** **1** **2**

(10 points) Suppose we have a function g defined (for n a power of 2) by

$$\begin{aligned} g(1) &= 3 \\ g(n) &= 4g(n/2) + n \text{ for } n \geq 2 \end{aligned}$$

Your partner has already figured out that

$$g(n) = 4^k g(n/2^k) + \sum_{p=0}^{k-1} n2^p$$

Finish finding the closed form for $g(n)$ assuming that n is a power of 2. Show your work and simplify your answer. Recall that $\log_b n = (\log_a n)(\log_b a)$.

Solution: To find the value of k at the base case, set $n/2^k = 1$. Then $n = 2^k$, so $k = \log_2 n$. Notice also that $4^{\log_2 n} = 4^{\log_4 n \log_4 2} = n^2$.

Substituting this into the above, we get

$$\begin{aligned} g(n) &= 4^k g(n/2^k) + n \sum_{p=0}^{k-1} 2^p \\ &= 4^{\log_2 n} \cdot 3 + n \sum_{p=0}^{\log_2 n - 1} 2^p \\ &= 3n^2 + n(2^{\log_2 n} - 1) \\ &= 3n^2 + n(n-1) \end{aligned}$$

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1. (8 points) Suppose we have a function g defined (for n a power of 3) by

$$\begin{aligned} g(9) &= 5 \\ g(n) &= 3g(n/3) + n \text{ for } n \geq 27 \end{aligned}$$

Your partner has already figured out that

$$g(n) = 3^k g(n/3^k) + kn$$

Finish finding the closed form for g. Show your work and simplify your answer.

Solution:

To find the value of k at the base case, set $n/3^k = 9$. Then $n = 3^{k+2}$, so $k+2 = \log_3 n$, so $k+2 = \log_3 n - 2$ Substituting this into the above, we get:

$$\begin{aligned} g(n) &= 3^{log_3 n - 2} g(9) + (log_3 n - 2)n \\ &= 3^{log_3 n} 3^{-2} 5 + n \log_3 n - 2n) \\ &= \frac{5}{9} n + n \log_3 n - 2n = n \log_3 n - \frac{15}{9} n \end{aligned}$$

2. (2 points) Check the (single) box that best characterizes each item.

The number of nodes in the 4-dimensional hypercube Q_4	4 ☐	16 ☑	32 ☐	64 ☐

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Discussion: **Thursday 2 3 4 5 Friday 9 10 11 12 1 2**

1. (8 points) Suppose we have a function g defined (for n a power of 2) by

$$\begin{aligned} g(1) &= c \\ g(n) &= 4g(n/2) + n \text{ for } n \geq 2 \end{aligned}$$

Express $g(n)$ in terms of $g(n/2^3)$ (where $n \geq 8$). Show your work and simplify your answer.

Solution:

$$\begin{aligned} g(n) &= 4g(n/2) + n \\ &= 4(4g(n/2^2 + n/2) + n \\ &= 4(4(4g(n/2^3) + n/2^2) + n/2) + n \\ &= 4^3 g(n/2^3) + n(2^2 + 2 + 1) \\ &= 4^3 g(n/2^3) + 7n \end{aligned}$$

2. (2 points) Suppose that $f : \mathbb{N} \to \mathbb{N}$ is such that $f(n) = n!$. Give a recursive definition of f

Solution:

$f(0) = 1$, and $f(n) = nf(n-1)$ for $n \geq 1$.

You could also have used $f(n+1) = (n+1)f(n)$ for $n \geq 0$.

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(10 points) Suppose we have a function F defined (for n a power of 3) by

$$\begin{aligned} F(1) &= 5 \\ F(n) &= 3F(n/3) + 7 \text{ for } n \geq 3 \end{aligned}$$

Your partner has already figured out that

$$F(n) = 3^k F(n/3^k) + 7\sum_{p=0}^{k-1} 3^p$$

Finish finding the closed form for F. Show your work and simplify your answer. Recall the following useful closed form (for $r \neq 1$): $\displaystyle\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$

Solution:

To find the value of k at the base case, set $n/3^k = 1$. Then $n = 3^k$, so $k = \log_3 n$. Substituting this into the above, we get

$$\begin{aligned} F(n) &= 3^{\log_3 n} \cdot 5 + 7 \sum_{p=0}^{\log_3 n - 1} 3^p \\ &= 5n + 7\frac{3^{log_3 n} - 1}{3-1} \\ &= 5n + 7\frac{n-1}{3-1} = 5n + \frac{7(n-1)}{2} \end{aligned}$$

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1. (8 points) Suppose we have a function f defined by

$$\begin{aligned} f(0) &= f(1) = 3 \\ f(n) &= 5f(n-2) + d, \text{ for } n \geq 2 \end{aligned}$$

where d is a constant. Express $f(n)$ in terms of $f(n-6)$ (where $n \geq 6$). Show your work and simplify your answer.

Solution:

$$\begin{aligned} f(n) &= 5f(n-2) + d \\ &= 5(5(f(n-4) + d) + d) \\ &= 5(5(5(f(n-6) + d) + d) + d) \\ &= 5^3 f(n-6) + (25 + 5 + 1)d \\ &= 5^3 f(n-6) + 31d \end{aligned}$$

2. (2 points) Suppose that G_0 is the graph consisting of a single vertex. Also suppose that the graph G_n consists of a copy of G_{n-1} plus an extra vertex v and edges joining v to each vertex in G_{n-1}. Give a clear picture or precise description of G_4.

Solution: This is a recursive construction of all the complete graphs, except for the indexing being off by one. So G_4 is just K_5.

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(10 points) Suppose we have a function g defined (for n a power of 4) by

$$\begin{aligned} g(1) &= c \\ g(n) &= 2g(n/4) + n \text{ for } n \geq 4 \end{aligned}$$

Your partner has already figured out that

$$g(n) = 2^k g(n/4^k) + n \sum_{p=0}^{k-1} \frac{1}{2^p}$$

Finish finding the closed form for $f(n)$ assuming that n is a power of 4. Show your work and simplify your answer. Recall that $\log_b n = (\log_a n)(\log_b a)$.

Solution: To find the value of k at the base case, set $n/4^k = 1$. Then $n = 4^k$, so $k = \log_4 n$. Notice also that $2^{\log_4 n} = 2^{\log_2 n \log_2 4} = n^{1/2} = \sqrt{n}$

Substituting this into the above, we get

$$\begin{aligned} g(n) &= 2^{\log_4 n} \cdot c + n \sum_{p=0}^{\log_4 n - 1} \frac{1}{2^p} \\ &= 2^{\log_4 n} \cdot c + n(2 - \frac{1}{2^{\log_4 n - 1}}) \\ &= c\sqrt{n} + n(2 - \frac{2}{\sqrt{n}}) \\ &= c\sqrt{n} + 2n - 2\sqrt{n} \\ &= 2n - (c-2)\sqrt{n} \end{aligned}$$