CS 173, Fall 2015
Examlet 8, Part A

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FIRST: **LAST:**

Discussion: **Thursday 2 3 4 5 Friday 9 10 11 12 1 2**

(20 points) Use (strong) induction to prove that $a - b$ divides $a^n - b^n$, for any integers a and b and any natural number n.

Hint: $(a^n - b^n)(a + b) = (a^{n+1} - b^{n+1}) + ab(a^{n-1} - b^{n-1})$, for any real numbers a and b.

Solution: Let a and b be integers.

Proof by induction on n.

Base case(s):

At $n = 0$, $a^n - b^n = 1 - 1 = 0$, which is a multiple of any integer. So it's divisible by $a - b$.

At $n = 1$, $a^n - b^n = a - b$, so $a - b$ divides $a^n - b^n$.

[Notice that we need two base cases because our inductive step will use the result at two previous values of n.]

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $a - b$ divides $a^n - b^n$, for $n = 0, 1, \ldots, k$.

Rest of the inductive step:

From the hint, we know that $a^{k+1} - b^{k+1} = (a^k - b^k)(a + b) - ab(a^{k-1} - b^{k-1})$

Notice that $(a^k - b^k)$ is divisible by $(a - b)$ by the inductive hypothesis. $(a + b)$ is an integer since a and b are integers. So $(a^k - b^k)(a + b)$ must be divisible by $(a - b)$.

Similarly, $(a^{k-1} - b^{k-1})$ is divisible by $(a - b)$ by the inductive hypothesis. Also ab is an integer because a and b are integers. So $ab(a^{k-1} - b^{k-1})$ is divisible by $(a - b)$.

So $(a^k - b^k)(a + b) - ab(a^{k-1} - b^{k-1})$ must be divisible by $(a - b)$, and therefore $a^{k+1} - b^{k+1}$ must be divisible by $(a - b)$, which is what we needed to show.

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(20 points) Recall that the hypercube Q_1 consists of two nodes joined by an edge and that Q_n consists of two copies of Q_{n-1} plus edges connecting corresponding nodes. Use (strong) induction to show Q_n has chromatic number 2 for any natural number $n \geq 1$.

Solution:

Proof by induction on n.

Base case(s): At $n = 1$, Q_1 must have chromatic number 2 since it has only two nodes but they can't share the same color since they are joined by an edge.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that Q_n has chromatic number 2 for $n = 1, 2, \ldots, k-1$.

Rest of the inductive step: Consider Q_k. It must have chromatic number at least 2, since it contains edges. We need to show that it can actually be colored with 2 colors.

Q_k consists of two copies of Q_{k-1}, plus the connecting edges of the form xx' where x and x' are corresponding nodes in the two copies. By the inductive hypothesis, we can color the first Q_{k-1} with two colors. Color the other copy with the same two colors, but swapped. The connecting edges xx' join edges of opposite colors, so we now have a coloring of the whole graph Q_k.

So Q_k has chromatic number 2, which is what we needed to show.

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The original had the hint written as $(a^n - b^n)(a + b) = (a^{n+1} - b^{n+1}) + ab(a^{n-1} - b^{n-1})$, for any real numbers a and b. The following uses the intended hint (partly so the bug is fixed for the future). However, we're taking this issue into account when grading.

(20 points) Use (strong) induction to prove that $(3 + \sqrt{5})^n + (3 - \sqrt{5})^n$ is an integer for all natural numbers n

Hint: $(a^n + b^n)(a + b) = (a^{n+1} = b^{n+1}) + ab(a^{n-1} + b^{n-1})$, for any real numbers a and b.

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $(3 + \sqrt{5})^n + (3 - \sqrt{5})^n = 1 + 1 = 2$, which is an integer.

At $n = 1$, $(3 + \sqrt{5})^n + (3 - \sqrt{5})^n = (3 + \sqrt{5}) + (3 - \sqrt{5}) = 6$, which is an integer.

[Notice that we need two base cases because our inductive step will use the result at two previous values of n.]

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is an integer for $n = 0, 1, \dots, k$.

Rest of the inductive step:

Let $a = (3 + \sqrt{5})$ and $a = (3 - \sqrt{5})$. Then the inductive hypothesis tells us that $a^k + b^k$ is an integer, and $a^{k-1} + b^{k-1}$ is an integer.

Notice also that $a + b = 6$ and $ab = (3 + \sqrt{5})(3 - \sqrt{5}) = 9 - 5 - 4$.

Now, using the hint, we can calculate

$$
\begin{aligned}
(3 + \sqrt{5})^{k+1} + (3 - \sqrt{5})^{k+1} &= a^{k+1} + b^{k+1} \\
&= (a^k + b^k)(a + b) - ab(a^{k-1} + b^{k-1}) \\
&= 6(a^k + b^k) - 4(a^{k-1} + b^{k-1})
\end{aligned}
$$

The righthand expression $6(a^k+b^k)-4(a^{k-1}+b^{k-1})$ must be an integer because it's made by multiplying and subtracting integers. So the lefthand expression, i.e. $(3 + \sqrt{5})^{k+1} + (3 - \sqrt{5})^{k+1}$ must be an integer, whcih is what we needed to show.

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(20 points) (20 points) Suppose that $f : \mathbb{N} \to \mathbb{Z}$ is defined by

$f(0) = 2 \qquad f(1) = 5 \qquad f(2) = 15$

$f(n) = 6f(n-1) - 11f(n-2) + 6f(n-3)$, for all $n \geq 2$

Use (strong) induction to prove that $f(n) = 1 - 2^n + 2 \cdot 3^n$

Solution:

Proof by induction on n.

Base case(s): At $n = 0$, $f(0) = 2$ and $1 - 2^n + 2 \cdot 3^n = 1 - 1 + 2 = 2$

At $n = 1$, $f(1) = 5$ and $1 - 2^n + 2 \cdot 3^n = 1 - 2 + 6 = 5$

At $n = 2$, $f(2) = 15$ and $1 - 2^n + 2 \cdot 3^n = 1 - 4 + 18 = 15$

So the claim holds at all three values.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $f(n) = 1 - 2^n + 2 \cdot 3^n$ for $n = 0, 1, \ldots, k-1$.

Rest of the inductive step: By the definition of f and the inductive hypothesis, we get

$$\begin{aligned}
f(k) &= 6f(k-1) - 11f(k-2) + 6f(k-3) \\
&= 6(1 - 2^{k-1} + 2 \cdot 3^{k-1}) - 11(1 - 2^{k-2} + 2 \cdot 3^{k-2}) + 6(1 - 2^{k-3} + 2 \cdot 3^{k-3}) \\
&= (6 - 11 + 6) - (6 \cdot 2^{k-1} - 11 \cdot 2^{k-2} + 6 \cdot 2^{k-3}) + 2(6 \cdot 3^{k-1} - 11 \cdot 3^{k-2} + 6 \cdot 3^{k-3}) \\
&= 1 - (12 \cdot 2^{k-2} - 11 \cdot 2^{k-2} + 3 \cdot 2^{k-2}) + 2(18 \cdot 3^{k-2} - 11 \cdot 3^{k-2} + 2 \cdot 3^{k-2}) \\
&= 1 - 4 \cdot 2^{k-2} + 2 \cdot 9 \cdot 3^{k-2} = 1 - 2^k + 2 \cdot 2^k
\end{aligned}$$

So $f(k) = 1 - 2^k + 2 \cdot 2^k$, which is what we needed to show.

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(20 points) Recall that the hypercube Q_2 is a 4-cycle, and that Q_n consists of two copies of Q_{n-1} plus edges connecting corresponding nodes. A *Hamiltonian cycle* is a cycle that visits each node exactly once, except obviously for when it returns to the starting node at the end. Use (strong) induction to show Q_n has Hamiltonian cycle for any natural number $n \geq 2$.

Solution:

Proof by induction on n.

Base case(s): At $n = 2$, Q_2 is the same as (isomorphic to) C_4. The entire graph forms a Hamiltonian cycle.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that Q_n has a Hamiltonian cycle for $n = 2, 3, \ldots, k-1$.

Rest of the inductive step: Consider Q_k. Q_k consists of two copies of Q_{k-1}, plus the connecting edges of the form xx' where x and x' are corresponding nodes in the two copies. Each of the smaller hypercubes has a Hamiltonian cycle by the inductive hypothesis. Remove an edge ab from one of these cycles and the corresponding edge $a'b'$ from the other cycle. Next, join the two partial cycles using the connector edges aa' and bb' This new cycle is a Hamiltonian cycle for Q_k.

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(20 points) Let function $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$f(0) = 3$

$f(1) = 9$

$f(n) = f(n-1) + 2f(n-2)$, for $n \geq 2$

Use (strong) induction to prove that $f(n) = 4 \cdot 2^n + (-1)^{n-1}$ for any natural number n.

Solution: Proof by induction on n.

Base case(s): For $n = 0$, we have $4 \cdot 2^0 + (-1)^{-1} = 4 - 1 = 3$ which is equal to $f(0)$. So the claim holds.

For $n = 1$, we have $4 \cdot 2^1 + (-1)^0 = 8 + 1 = 9$ which is equal to $f(1)$. So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $f(n) = 4 \cdot 2^n + (-1)^{n-1}$ for $n = 0, 1, \ldots, k-1$ where $k \geq 2$.

Rest of the inductive step:

$$
\begin{aligned}
f(k) &= f(k-1) + 2f(k-2) && \text{by definition of } f \\
&= (4 \cdot 2^{k-1} + (-1)^{k-2}) \quad + \quad 2(4 \cdot 2^{k-2} + (-1)^{k-3}) && \text{by inductive hypothesis} \\
&= (4 \cdot 2^{k-1} + (-1)^{k-2}) \quad + \quad 4 \cdot 2^{k-1} + 2(-1)^{k-3} \\
&= 8 \cdot 2^{k-1} + (-1)^{k-2} - 2(-1)^{k-2} \\
&= 4 \cdot 2^k - (-1)^{k-2} \\
&= 4 \cdot 2^k + (-1)^{k-1}
\end{aligned}
$$

So $f(k) = 4 \cdot 2^k(-1)^{k-1}$, which is what we needed to show.