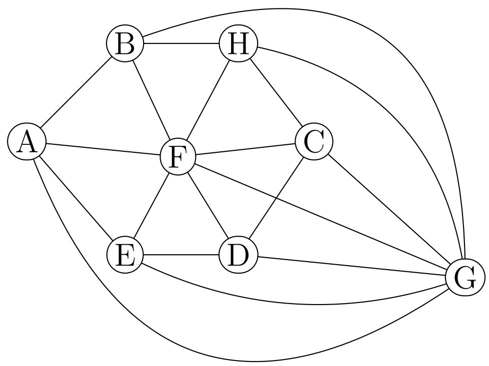
CS 173, Fall 2015
Examlet 7, Part B

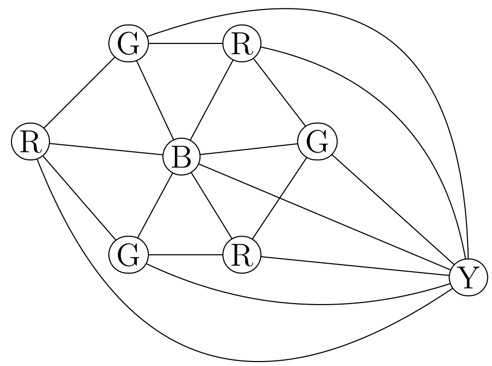
NETID:

FIRST: **LAST:**

Discussion: **Thursday** **2** **3** **4** **5** **Friday** **9** **10** **11** **12** **1** **2**

1. (9 points) What is the chromatic number of graph G (below)? Justify your answer.





Solution: The chromatic number is four. The picture above shows how to color it with four colors (upper bound).

For the lower bound, the graph contains a W_6 whose hub is F and whose rim contains nodes A, B, C, D, E, H. Coloring a W_6 requires three colors. Then the node G is connected to all seven nodes in the W_6, so it needs a different, fifth color.

2. (6 points) Check the (single) box that best characterizes each item.

Exactly 11 Xboxes fit in my suitcase by volume, but I haven't checked their total weight. 11 is ____ how many Xboxes the suitcase can hold.

an upper bound on	√	exactly	
a lower bound on		not a bound on	

All elements of M are also elements of X.

$M = X$		$M \subseteq X$	√	$X \subseteq M$	

$\sum_{i=1}^{p-1} i =$

$\frac{p(p-1)}{2}$	√	$\frac{(p-1)^2}{2}$		$\frac{p(p+1)}{2}$		$\frac{(p-1)(p+1)}{2}$	

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1. (11 points) Let's define two sets as follows:

$$A = \{x \in \mathbb{R} \ : \ |x+1| \leq 2\}$$
$$B = \{w \in \mathbb{R} \ : \ w^2 + 2w - 3 \leq 0\}$$

Prove that $A = B$ by proving two subset inclusions.

Solution: $A \subseteq B$: Let x be a real number and suppose $x \in A$. Then $|x+1| \leq 2$. Therefore, $-3 \leq x+1 \leq 1$. Therefore $x+3 \geq 0$ and $x-1 \leq 0$. So $x^2+2x-3 = (x+3)(x-1) \leq 0$. So $x \in B$.

$B \subseteq A$: Let x be a real number and suppose $x \in B$. Then $x^2 + 2x - 3 \leq 0$. Factoring this polynomial, we get $(x+3)(x-1) \leq 0$. So $(x+3)$ and $(x-1)$ must have opposite signs. Since $x+3 > x-1$, it must be the case that $x+3 \geq 0$ and $x-1 \leq 0$. Therefore, $-3 \leq x+1 \leq 1$. So $|x+1| \leq 2$, and therefore $x \in A$.

Since $A \subseteq B$ and $B \subseteq A$, $A = B$.

2. (4 points) Check the (single) box that best characterizes each item.

I found 143 identical marbles in my saucepan last Saturday. 143 is ____ how many marbles this size will fits in my saucepan.

an upper bound on	☐	exactly	☐
a lower bound on	☑ √	not a bound on	☐

Chromatic number of a bipartite graph with at least two vertices.

1	☐	2	☐	3	☐	can't tell	☑ √

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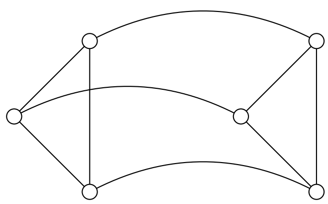
1. (11 points) Recall that if G is a graph, then $\chi(G)$ is its chromatic number. Let's define the "doubled" version of a graph G as follows: make two copies of G and add an edge joining each pair of corresponding nodes. For example, the doubled version of C_3 looks like:



Suppose that T is the doubled version of a graph G. Describe how $\chi(T)$ is related to $\chi(G)$, justifying your answer. Your answer should handle any choice for G, not just C_3.

Solution:

$\chi(T) = \max(2, \chi(G))$.

First, let's suppose that $\chi(G) \geq 2$. If $\chi(G) = n$, then we can start coloring T by coloring one copy of G with n colors. Let's call the colors $c_1, c_2, \ldots, c_n$. Now color the second copy of G using the rule that if a node in the first copy has color c_i, then the corresponding node in the second copy has color c_{i+1} if $i + 1 \leq n$ or c_1 if $i + 1 = n$. This shows that $\chi(T) = \chi(G)$.

This construction won't work if $\chi(G)$ is 1. In this case, there aren't any edges in G. So the only edges in T connect pairs of corresponding nodes. This means that T requires two colors.

2. (4 points) Check the (single) box that best characterizes each item.

All elements of X are also elements of M.	$M = X$ ☐	$M \subseteq X$ ☐	$X \subseteq M$ ☑ √

$\sum_{k=1}^{n} \frac{1}{2^k}$	$1 - (\frac{1}{2})^{n-1}$ ☐	$2 - (\frac{1}{2})^n$ ☐	$1 - (\frac{1}{2})^n$ ☑ √	$2 - (\frac{1}{2})^{n-1}$ ☐

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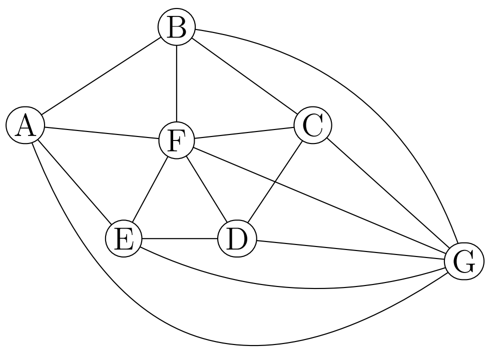
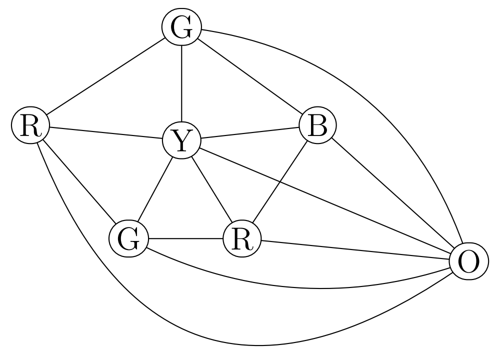
1. (9 points) What is the chromatic number of graph G (below)? Justify your answer.





Solution: The chromatic number is five. The picture above shows how to color it with five colors (upper bound).

For the lower bound, the graph contains a W_5 whose hub is F and whose rim contains nodes A, B, C, D, E. Coloring a W_5 requires four colors. Then the node G is connected to all six nodes in the W_5, so it needs a different, fifth color.

2. (6 points) Check the (single) box that best characterizes each item.

$\sum_{i=0}^{k-1}(k \cdot i + 2) =$

$\frac{k^2(k-1)}{2} + 2k$	☑ √	$\frac{k(k+1)}{2} + 2(k-1)$	☐
$\frac{k^2(k+1)}{2} + 2k$	☐	$\frac{k(k-1)}{2} + 2(k-1)$	☐

Putting 10 people in the canoe caused it to sink. 10 is _____ how many people the canoe can carry.

an upper bound on	☑ √	exactly	☐
a lower bound on	☐	not a bound on	☐

The chromatic number of a graph with maximum vertex degree D

$= D$	☐	$= D + 1$	☐
$\leq D + 1$	☑ √	$\geq D + 1$	☐

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1. (11 points) Let's define two sets as follows:

$$A = \{(p+1, p) \ : \ p \in \mathbb{R}\}$$
$$B = \{\lambda(1,0) + (1-\lambda)(2,1) \ : \ \lambda \in \mathbb{R}\}$$

Prove that $A = B$ by proving two subset inclusions.

Solution: $B \subseteq A$: Let (x, y) be a pair of real numbers such that $(x, y) \in B$. Then $(x, y) = \lambda(1,0) + (1-\lambda)(2,1)$ for some real number λ. Then $x = \lambda + 2 - 2\lambda = 2 - \lambda$ and $y = 1 - \lambda$. So $x = y + 1$. So (x, y) has the form $(p + 1, p)$ and therefore $(x, y) \in A$.

$A \subseteq B$: Let (x, y) be a pair of real numbers such that $(x, y) \in A$. Then $x = y + 1$. Consider $\lambda = 1 - y$. Then $y = 1 - \lambda$ and $x = 2 - \lambda = \lambda + 2(1 - \lambda)$. So $(x, y) = \lambda(1,0) + (1-\lambda)(2,1)$. Therefore $(x, y) \in A$.

Since $A \subseteq B$ and $B \subseteq A$, $A = B$.

2. (4 points) Check the (single) box that best characterizes each item.

Suppose I want to estimate $\frac{103}{20}$. 3 is _____

an upper bound	☐	an exact answer	☐
a lower bound	☑ √	not a bound on	☐

The chromatic number of C_n.

2 ☐	3 ☐	≤ 3 ☑ √	≤ 4 ☐

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1. (11 points) Recall that if G is a graph, then $\chi(G)$ is its chromatic number. Suppose that G is a graph and H is another graph not connected to G. Suppose G and H each have at least two nodes and at least one edge. Dr. Evil picks two adjacent nodes a and b from G, and also two adjacent nodes c and d from H. He merges G and H into a single graph T by merging b and d into a single node, and adding an edge connecting a and c. So, if G and H are as shown on the left, then T might look as shown on the right.

Describe how $\chi(T)$ is related to $\chi(G)$ and $\chi(H)$, justifying your answer.

Solution: $\chi(T) = \max(\chi(G), \chi(H), 3)$

The output graph contains a triangle, so it definitely requires at least three colors.

Without loss of generality, suppose that $k = \chi(G) \geq \chi(H)$. Then $\chi(T)$ must be at least k because G is a subgraph of T. Also notice that k is at least 2 because the two input graphs each contain an edge.

First, suppose k is at least 3. To color T with k colors, first color the part of T corresponding to G. We have a coloring of H that uses $\leq k$ colors, but the color choices might not be compatible with how we've started coloring T. If the two merged nodes b and d have different colors, swap the names of two colors to make them same. If a and c have the same color, swap the color of c with some third color, remembering that k is at least 3. Adjust the rest of the coloring for H to use these same choices of color names.

Special case: if $k = 2$, then we carry out the same procedure. However, we won't have any third color available to fix the color of c, so we'll have to allocate an extra color.

2. (4 points) Check the (single) box that best characterizes each item.

$\sum_{k=3}^{n} k^7 =$ $\sum_{p=1}^{n-2} p^9$ ☐ $\sum_{p=1}^{n-2} k^7$ ☐ $\sum_{p=1}^{n-2} k^9$ ☐ $\sum_{p=1}^{n-2} (p+2)^7$ ☑ (√)

W_7 is a subgraph of graph H. 4 is _____ the chromatic number of H.

an upper bound on	☐	exactly	☐
a lower bound on	☑ (√)	not a bound on	☐