CS 173, Fall 2015
Examlet 7, Part A

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Use (strong) induction to prove the following claim:

Claim: $(4n)!$ is divisible by 8^n, for all positive integers n.

Solution: Proof by induction on n.

Base case(s): At $n = 1$, the claim amounts to "4! is divisible by 8." $4! = 24$ which is clearly divisible by 8.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $(4n)!$ is divisible by 8^n, for $n = 1, 2, \ldots, k$.

Rest of the inductive step: At $n = k + 1$, $(4n)! = (4(k + 1))! = (4k + 4)! = (4k + 4)(4k + 3)(4k + 2)(4k + 1)(4k)!$

Now, $(4k + 4)$ is divisible by 4, and $(4k + 2)$ is divisible by 2. So $(4k + 4)(4k + 3)(4k + 2)(4k + 1)$ is divisible by 8. By the inductive hypothesis, we know that $(4k)!$ is divisible by 8^k. Combining these two facts, $(4(k + 1))!$ is divisible by 8^{k+1}, which is what we needed to show.

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Use (strong) induction to prove the following claim:

For all positive integers n, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $\sum_{i=1}^{n} i^2 = 1$ and $\frac{n(n+1)(2n+1)}{6} = \frac{1 \cdot 2 \cdot 3}{6} = 1$. So the claim is true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for $n = 1, 2, \ldots, k$.

Rest of the inductive step: By the inductive hypothesis, we know that $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= (\sum_{i=1}^{k} i^2) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= (k+1)\frac{k(2k+1)}{6} + (k+1) = (k+1)\frac{k(2k+1)+6k+6}{6}
\end{aligned}$$

But $k(2k+1) + 6k + 6 = 2k^2 + 7k + 6 = (n+2)(2n+3)$. So $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$ which is $\frac{n(n+1)(2n+1)}{6}$ at $n = k+1$. So the claim holds for $n = k+1$.

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Use (strong) induction to prove the following claim:

Claim: $\sum_{p=0}^{n}(p \cdot p!) = (n+1)! - 1$, for all natural numbers n.

Recall that 0! is defined to be 1.

Solution: Proof by induction on n.

Base case(s):

Solution: At $n = 0$, $\sum_{p=0}^{n}(p \cdot p!) = 0 \cdot 0! = 0$ Also $(n+1)! - 1 = 0! - 1 = 1 - 1 = 0$. So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Solution: Suppose that $\sum_{p=0}^{n}(p \cdot p!) = (n+1)! - 1$, for $n = 0, 1, \ldots, k$.

Rest of the inductive step:

Solution: By the inductive hypothesis $\sum_{p=0}^{k}(p \cdot p!) = (k+1)! - 1$. So

$$\begin{aligned}
\sum_{p=0}^{k+1}(p \cdot p!) &= ((n+1) \cdot (n+1)!) + \sum_{p=0}^{k}(p \cdot p!) \\
&= ((k+1) \cdot (k+1)!) + \sum_{p=0}^{k}(p \cdot p!) \\
&= (n+1) \cdot (k+1)! + (k+1)! - 1 \\
&= (k+1) \cdot (k+1)! + (k+1)! - 1 \\
&= [(k+1)+1] \cdot (k+1)! - 1 \\
&= (k+2) \cdot (k+1)! - 1 = (k+2)! - 1
\end{aligned}$$

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Use (strong) induction to prove the following claim:

Claim: for all natural numbers n, $\displaystyle\sum_{j=0}^{n} 2(-7)^j = \frac{1-(-7)^{n+1}}{4}$

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $\sum_{j=0}^{n} 2(-7)^j = 2$ and $\frac{1-(-7)^{n+1}}{4} = \frac{1-(-7)}{4} = 2$. So the claim holds at $n = 0$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\displaystyle\sum_{j=0}^{n} 2(-7)^j = \frac{1-(-7)^{n+1}}{4}$ for $n = 0, 1, \ldots, k$.

Rest of the inductive step:

In particular $\displaystyle\sum_{j=0}^{k} 2(-7)^j = \frac{1-(-7)^{k+1}}{4}$. So then

$$\begin{aligned}
\sum_{j=0}^{k+1} 2(-7)^j &= (\sum_{j=0}^{n} 2(-7)^j) + 2(-7)^{k+1} \\
&= \frac{1-(-7)^{k+1}}{4} + 2(-7)^{k+1} = \frac{1-(-7)^{k+1}+8(-7)^{k+1}}{4} = \frac{1+7(-7)^{k+1}}{4} \\
&= \frac{1-(-7)^{k+2}}{4}
\end{aligned}$$

So $\displaystyle\sum_{j=0}^{k+1} 2(-7)^j = \frac{1-(-7)^{k+2}}{4}$, which is what we needed to show.

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Use (strong) induction and the fact that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ to prove the following claim:

For all natural numbers n, $\left(\sum_{i=0}^{n} i\right)^2 = \sum_{i=0}^{n} i^3$

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $\left(\sum_{i=0}^{n} i\right)^2 = 0^2 = 0 = \sum_{i=0}^{n} i^3$. So the claim is true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:
Suppose that $\left(\sum_{i=0}^{n} i\right)^2 = \sum_{i=0}^{n} i^3$ for $n = 0, 1, \ldots, k$.

Rest of the inductive step:
Starting with the lefthand side of the equation for $n = k+1$, we get

$$\left(\sum_{i=0}^{k+1} i\right)^2 = \left((k+1) + \sum_{i=0}^{k} i\right)^2 = (k+1)^2 + 2(k+1)\sum_{i=0}^{k} i + \left(\sum_{i=0}^{k} i\right)^2$$

By the inductive hypothesis $\left(\sum_{i=0}^{k} i\right)^2 = \sum_{i=0}^{k} i^3$. Substituting this and the fact we were told to assume, we get

$$\left(\sum_{i=0}^{k+1} i\right)^2 = (k+1)^2 + 2(k+1)\frac{k(k+1)}{2} + \sum_{i=0}^{k} i^3 = (k+1)^2 + k(k+1)^2 + \sum_{i=0}^{k} i^3 = (k+1)^3 + \sum_{i=0}^{k} i^3 = \sum_{i=0}^{k+1} i^3$$

So $\left(\sum_{i=0}^{k+1} i\right)^2 = \sum_{i=0}^{k+1} i^3$ which is what we needed to show.

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Use (strong) induction to prove the following claim:

For all positive integers n, $\sum_{p=1}^{n}(-1)^{p-1}p^2 = \frac{(-1)^{n-1}n(n+1)}{2}$

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $\sum_{p=1}^{n}(-1)^{p-1}p^2 = (-1)^0 \cdot 0^2 = 0$.
And $\frac{(-1)^{n-1}n(n+1)}{2} = \frac{(-1)^{-1}0\cdot 1}{2} = 0$. So the claim holds at $n = 1$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:
$\sum_{p=1}^{n}(-1)^{p-1}p^2 = \frac{(-1)^{n-1}n(n+1)}{2}$ for $n = 1, 2, \cdot, k$.

Rest of the inductive step:

By the inductive hypothesis, $\sum_{p=1}^{k}(-1)^{p-1}p^2 = \frac{(-1)^{k-1}k(k+1)}{2}$ for

$$\begin{aligned}
\sum_{p=1}^{k+1}(-1)^{p-1}p^2 &= (-1)^k(k+1)^2 + \sum_{p=1}^{k}(-1)^{p-1}p^2 = (-1)^k(k+1)^2 + \frac{(-1)^{k-1}k(k+1)}{2} \\
&= (-1)^k(k+1)^2 - \frac{(-1)^k k(k+1)}{2} = (-1)^k(k+1)\left((k+1) - \frac{k}{2}\right) \\
&= (-1)^k(k+1)\frac{2(k+1)-k}{2} = \frac{(-1)^k(k+1)(k+2)}{2}
\end{aligned}$$

So $\sum_{p=1}^{k+1}(-1)^{p-1}p^2 = \frac{(-1)^k(k+1)(k+2)}{2}$ which is the claim at $n = k+1$ i.e. what we needed to show.