CS 173, Fall 2015
Examlet 5, Part A

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Discussion: **Thursday** **2** **3** **4** **5** **Friday** **9** **10** **11** **12** **1** **2**

1. (10 points) Suppose that $f : A \to B$ and $g : B \to C$ are onto. Prove that $g \circ f$ is onto.

   Solution: Let z be an element of C. Since g is onto, there is a $y \in B$ such that $g(y) = z$. Since f is onto, there is an $x \in A$ such that $f(x) = y$. But then $g \circ f(x) = g(f(x)) = g(y) = z$. So z has a pre-image.

   Since we found a pre-image for an arbitrarily-chosen element of C, $g \circ f$ is onto.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : \mathbb{R} \to \mathbb{R}$ to be "strictly increasing." You must use explicit quantifiers.

   Solution: For all x and y in $\mathbb{R}$, if $x < y$ then $g(x) < g(y)$.

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1. (10 points) Suppose that $f : A \to B$ and $g : B \to C$ are one-to-one. Prove that $g \circ f$ is one-to-one.

   Solution: Let x and y be elements of A and suppose that $g \circ f(x) = g \circ f(y)$. That is $g(f(x)) = g(f(y))$. Since g is one-to-one, this implies that $f(x) = f(y)$. Since f is one-to-one, this implies that $x = y$.

   We've shown that $g \circ f(x) = g \circ f(y)$ implies $x = y$ for any x and y in A. So $g \circ f$ is one-to-one.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : C \to M$ to be "onto." You must use explicit quantifiers. Do not assume the reader knows what the image of the function is.

   Solution: For every element y in M, there is an element x in C such that $g(x) = y$.

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1. (10 points) Suppose that $g : \mathbb{N} \to \mathbb{N}$ is one-to-one. Let's define the function $f : \mathbb{N}^2 \to \mathbb{N}^2$ by the equation $f(x, y) = (x + g(y), g(x))$. Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) the behavior of increasing functions.

   Solution: Let (x, y) and (a, b) be pairs of natural numbers and suppose that $f(x, y) = f(a, b)$.

   By the definition of f, we know that $x + g(y) = a + g(b)$ and $g(x) = g(a)$.

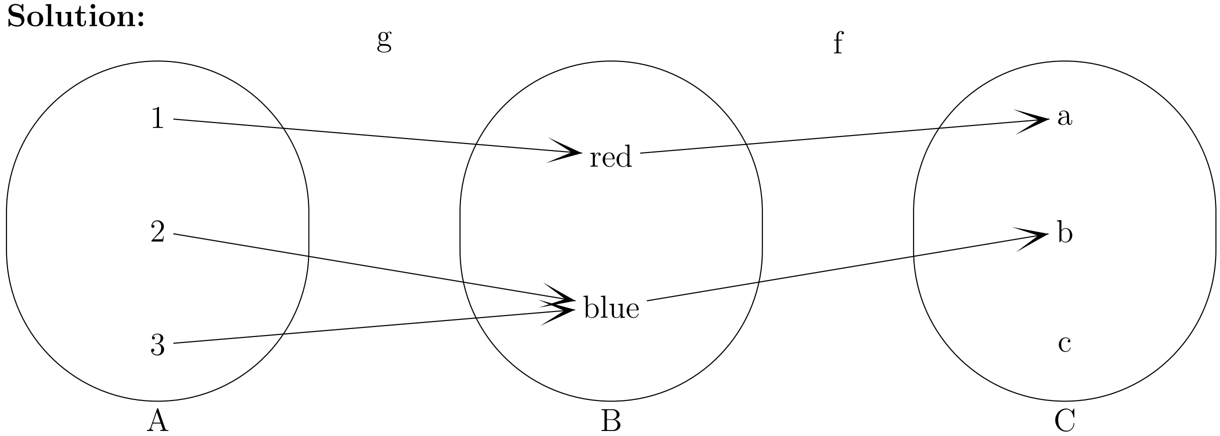
   Since g is one-to-one and $g(x) = g(a)$, $x = a$. Substituting this into $x + g(y) = a + g(b)$, we get $x + g(y) = x + g(b)$, so $g(y) = g(b)$.

   Since g is one-to-one, $g(y) = g(b)$ implies that $y = b$.

   Since $x = a$ and $y = b$, $(x, y) = (a, b)$, which is what we needed to show.

2. (5 points) Suppose that $g : A \to B$ and $f : B \to C$. Prof. Snape claims that if g is onto, then $f \circ g$ is onto. Disprove this claim using a concrete counter-example in which A, B, and C are all small finite sets.

   Solution:



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1. (10 points)

   Note: the first proof had the domains and co-domains of the functions mistakenly given in terms of the natural numbers rather than the integers. Almost everyone did the problem as intended (below) rather than as originally written. We did something sensible grading the few people who did see the issue.

   Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is onto. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}$ by $g(x, y) = (1 - f(x))f(y)$. Prove that g is onto.

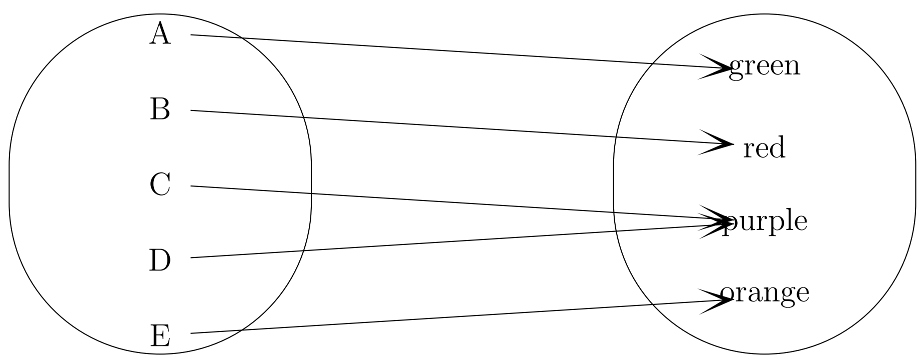
   Solution: Suppose that n is a natural number.

   Since f is onto, there is a natural number p such that $f(p) = 0$. Then $(1 - f(p)) = 1$

   Also since f is onto, there is a natural number q such that $f(q) = n$.

   Now consider the pair (p, q). $g(p, q) = (1 - f(p))f(q) = 1 \cdot n = n$. So (p, q) is a pre-image for n, which is what we needed to find.

2. (5 points) Complete this picture to make an example of a function that is onto but not one-to-one, by adding elements to the domain and arrows showing how input values map to output values. The elements of the domain must be letters of the alphabet.



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1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let's define $g\mathbb{Z} \to \mathbb{Z}^2$ by $g(n) = (|n|, f(n)|n|)$. Prove that g is one-to-one.

   Solution:

   Let p and q be integers. Suppose that $g(p) = g(q)$.

   By the definition of g, this means that $(|p|, f(p)|p|) = (|q|, f(q)|q|)$. So $|p| = |q|$ and $f(p)|p| = f(q)|q|$.

   Case 1: $|p| = 0$. Then $p = q = 0$. So $p = q$.

   Case 2: $|p|$ is non-zero. Substituting the first equation into the second, we get that $f(p)|p| = f(q)|p|$. So $f(p) = f(q)$. Since f is one-to-one, this means that $p = q$.

   So we've shown that $g(p) = g(q)$ implies that $p = q$, which means that g is one-to-one.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : \mathbb{R} \to \mathbb{R}$ to be "increasing." You must use explicit quantifiers.

   Solution: For all x and y in $\mathbb{R}$, if $x \leq y$ then $g(x) \leq g(y)$.

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1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is onto. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}$ by $g(x, y) = f(x - 7)f(y)$. Prove that g is onto.

   Solution: Suppose that n is an integer.

   Since f is onto, there is an integer p such that $f(p) = 1$. Let $x = p + 7$. Then $f(x - 7) = f(p) = 1$.

   Also since f is onto, there is a natural number y such that $f(y) = n$.

   Now consider the pair (x, y). $g(x, y) = f(x - 7)f(y) = 1 \cdot n = n$. So (x, y) is a pre-image for n, which is what we needed to find.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : C \to M$ to be "one-to-one." You must use explicit quantifiers; do not use words like "unique".

   Solution: For every elements x and y in C, if $g(x) = g(y)$, then $x = y$