CS 173, Fall 2015
Examlet 4, Part A

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Discussion: Thursday 2 3 4 5 Friday 9 10 11 12 1 2

Let T be the relation defined on $\mathbb{N}^2$ by

$(x, y)T(p, q)$ if and only if $x \leq p$ or $(x = p$ and $y \leq q)$

Prove that T is transitive.

Solution:

Let (x, y), (p, q) and (m, n) be pairs of natural numbers. Suppose that $(x, y)T(p, q)$ and $(p, q)T(m, n)$. By the definition of T, $(x, y)T(p, q)$ means that $x \leq p$ or $(x = p$ and $y \leq q)$. Similarly $(p, q)T(m, n)$ implies that $p \leq m$ or $(p = m$ and $q \leq n)$.

There are four cases:

Case 1: $x \leq p$ and $p \leq m$. Then $x \leq m$.

Case 2: $x \leq p$ and $p = m$. Then $x \leq m$.

Case 3: $x = p$ and $p \leq m$. Then $x \leq m$.

Case 4: $x = p$ and $p = m$. In this case, we must also have $y \leq q$ and $q \leq n$. So $x = m$ and $y \leq n$.

In all four cases, $(x, y)T(m, n)$, which is what we needed to show.

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The closed interval $[a, b]$ is defined by $[a, b] = \{x \in \mathbb{R} \ : \ a \leq x \leq b\}$. Let J be the set containing all closed intervals $[a, b]$. Let's define the relation F on J as follows:

$[s, t]F[p, q]$ if and only if $q \leq s$

Prove that F is antisymmetric.

Solution: Let $[s, t]$ and $[p, q]$ be two closed intervals. Suppose that $[s, t]F[p, q]$ and $[p, q]F[s, t]$.

By the definition of F, this means that $q \leq s$ and $t \leq p$. By the definition of closed interval, $s \leq t$ and $p \leq q$. So we have

$$p \leq q \leq s \leq t \leq p$$

So $p = q = s = t$ and therefore $[s, t] = [p, q]$, which is what we needed to show.

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Let $A = \mathbb{N} \times \mathbb{N}$, i.e. pairs of natural numbers.

Define a relation $\gg$ on A as follows:

$$(x, y) \gg (p, q) \text{ if and only if there exists an integer } n \geq 1 \text{ such that } (x, y) = (np, nq).$$

Prove that $\gg$ is antisymmetric.

Solution: Let (x, y) and (p, q) be pairs of natural numbers and suppose that $(x, y) \gg (p, q)$ and $(p, q) \gg (x, y)$.

By the definition of $\gg$, $(x, y) = (np, nq)$ and $(p, q) = m(x, y)$, for some positive integers m and n. So $x = np$, $y = nq$, $p = mx$ and $q = my$.

Combining these equations, we get $x = n(mx) = (nm)x$ and $y = n(my) = (nm)y$. So $nm = 1$. But this means that $n = m = 1$ since n and m are positive integers. So $x = p$ and $y = q$. So $(x, y) = (p, q)$, which is what we needed to show.

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Let T be the relation defined on $\mathbb{Z}^2$ by

$(x, y)T(p, q)$ if and only if $x < p$ or ($x = p$ and $y \leq q$)

Prove that T is antisymmetric.

Solution:

Let (x, y) and (p, q) be pairs of integers. Suppose that $(x, y)T(p, q)$ and $(p, q)T(x, y)$. By the definition of T $(x, y)T(p, q)$ means that $x < p$ or ($x = p$ and $y \leq q$). Similarly, $(p, q)T(x, y)$ means that $p < x$ or ($p = x$ and $q \leq y$).

There are four cases:

Case 1: $x < p$ and $p < x$. This is impossible.

Case 2: $x < p$ and $p = x$ and $q \leq y$. Also impossible.

Case 3: $p < x$ and $x = p$ and $y \leq q$. Impossible as well.

Case 4: $x = p$ and $y \leq q$ and $p = x$ and $q \leq y$. Since $y \leq q$ and $q \leq y$, $x = y$. So we have $(x, y) = (p, q)$.

$(x, y) = (p, q)$ is true, which is what we needed to show.

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Suppose that n is some positive integer. Let's define the relation R_n on the integers such that aR_nb if and only if $a \equiv b+1 \pmod{n}$. Prove the following claim

Claim: For any integers x, y, and z, if xR_ny and yR_nz and xR_nz, then $n = 1$.

You must work directly from the definition of congruence mod k, using the following version of the definition: $x \equiv y \pmod{k}$ iff $x - y = mk$ for some integer m. You may use the following fact about divisibility: for any non-zero integers p and q, if $p \mid q$, then $|p| \leq |q|$.

Solution: Let n be a positive integer and suppose that R is as defined above. Also x, y, and z be integers and suppose that xR_ny and yR_nz and xR_nz.

By the definition of R, this means that $x \equiv y+1 \pmod{n}$, $y \equiv z+1 \pmod{n}$, and $x \equiv z+1 \pmod{n}$.

Then $x - (y+1) = kn$ $y - (z+1) = jn$, and $x - (z+1) = pn$, for some integers k, j, and p.

So then $x = y + 1 + kn$, $y = z + 1 + jn$ and $x = z + 1 + pn$. So $x = z + 2 + kn + jn$. So $z + 1 + pn = z + 2 + kn + jn$. So $pn = 1 + kn + jn$. So $(p - k - j)n = 1$.

We know that $p - k - j$ is an integer, so $(p - k - j)n = 1$ implies that $n|1$. Therefore $|n| \leq 1$. But n is known to be a positive integer. So n must equal 1.

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A closed interval of the real line can be represented as a pair (c, r), where c is the center of the interval and r is its radius. Let $X = \{(c, r) \mid c, r \in \mathbb{R}, r \geq 0\}$ be the set of closed intervals represented this way.

Now, let's define the interval containment $\preceq$ on X as follows

$(c, r) \preceq (d, q)$ if and only if $r \leq q$ and $|c - d| + r \leq q$.

Prove that $\preceq$ is antisymmetric.

Solution: Let (c, r) and (d, q) be elements of X. Suppose that $(c, r) \preceq (d, q)$ and $(d, q) \preceq (c, r)$.

By the definition of $\preceq$, $(c, r) \preceq (d, q)$ means that $r \leq q$ and $|c - d| + r \leq q$. Similarly, $(d, q) \preceq (c, r)$ means that $q \leq r$ and $|d - c| + q \leq r$.

Since $r \leq q$ and $q \leq r$, $q = r$. Substituting this into $|c - d| + r \leq q$, we get $|c - d| + r \leq r$. So $|c - d| \leq 0$. Since the absolute value of a real number cannot be negative, this means that $|c - d| = 0$, so $c = d$.

Since $q = r$ and $c = d$, $(c, r) = (d, q)$, which is what we needed to prove.