CS 173, Fall 2015
Examlet 2, Part A

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Recall that a real number p is rational if there are integers m and n (n non-zero) such that $p = \frac{m}{n}$. Use this definition and your best mathematical style to prove the following claim by contrapositive.

> For all real numbers x and y, if x is not rational, then $2x + 3y$ is not rational or y is not rational.

You must begin by explicitly stating the contrapositive of the claim:

Solution: Let's prove the contrapositive. That is, for all real numbers x and y, if $2x+3y$ is rational and y is rational, then x is rational.

Let x and y be real numbers. Suppose that $2x + 3y$ is rational and y is rational. Then $2x + 3y = \frac{a}{b}$ and $y = \frac{m}{n}$, where a, b, m, n are integers, b and n non-zero.

Then $2x + 3\frac{m}{n} = \frac{a}{b}$

So $2x = \frac{a}{b} - \frac{3m}{n} = \frac{an-3bm}{bn}$

So $x = \frac{an-3bm}{2bn}$

$an - 3bm$ and $2bn$ are both integers because a, b, m, n are integers. Also $2bn$ is non-zero because b and n are non-zero. So x is rational.

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Prove the following claim, using your best mathematical style and the following definition of congruence mod k: $a \equiv b \pmod{k}$ if and only if $a = b + nk$ for some integer n.

Claim: For all integers a, b, c, d, and k (k positive), if $a \equiv b \pmod{k}$ and $c \equiv d \pmod{k}$ then $a^2 + c \equiv b^2 + d \pmod{k}$.

Solution:

Let a, b, c, d, and k be integers, with k positive. Suppose that $a \equiv b \pmod{k}$ and $c \equiv d \pmod{k}$.

By the definition of congruence mod k, $a \equiv b \pmod{k}$ implies that $a = b + nk$ for some integer n. Similarly, $c \equiv d \pmod{k}$ implies that $c = d + mk$ for some integer m. Then we can calculate

$$a^2 + c = (b + nk)^2 + (d + mk) = b^2 + 2bnk + n^2k^2 + d + mk = b^2 + d + k(2bn + n^2k + m)$$

If we let $p = 2bn + n^2k + m$, then we have $a^2 + c = (b^2 + d) + kp$. Also, p must be an integer since b, n, k, and m are integers. So, by the definition of congruence mod k, $a^2 + c \equiv b^2 + d \pmod{k}$.

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Prove the following claim, using your best mathematical style and the following definition of congruence mod k: $a \equiv b \pmod{k}$ if and only if $a = b + nk$ for some integer n.

Claim: For all integers a, b, c, p, q where p and q are positive, if $a \equiv 2b+1 \pmod{p}$ and $c \equiv b \pmod{q}$ and $q \mid p$, then $a - 2c \equiv 1 \pmod{q}$.

Solution: Let a, b, c, p, q be integers with p and q positive. Suppose that $a \equiv 2b+1 \pmod{p}$ and $c \equiv b \pmod{q}$ and $q \mid p$. By the given definition of congruence, $a = (2b+1) + pr$ and $c = b + qt$, where r and t are integers. Since $q \mid p$, we know that $p = qu$, where u is an integer.

Therefore, by substituting $2b + 1 + pr$ for a and $b + qt$ for c, we get

$$a - 2c = 2b + 1 + pr - 2(b + qt)$$

By substituting qu for p, we get:

$a - 2c = 2b + 1 + qur - 2(b + qt) = 2b + 1 + qur - 2b - 2qt = 1 + q(ur - 2t) = 1 + qw$

where $w = ur - 2t$.

w must be an integer, because u, r, and t are integers. Therefore, by the definition given for congruence, $a - 2c \equiv 1 \pmod{q}$.

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Prove the following claim, using your best mathematical style. Hint: look at remainders and use proof by cases.

Prove that if n is an integer, then $n^2 + 2$ is not divisible by 4.

Solution: Let n be an integer. From the Division Algorithm (aka definition of remainder), we know that there are integers q and r such that $n = 4q + r$.

There are five cases, depending on what the remainder r is:

Case 1: $n = 4q$. Then $n^2 + 2 = 16q^2 + 2 = 4(4q^2) + 2$.

Case 2: $n = 4q + 1$. Then $n^2 + 2 = 16q^2 + 8q + 3 = 4(4q^2 + 2q) + 3$.

Case 3: $n = 4q + 2$. Then $n^2 + 2 = 16q^2 + 16q + 6 = 4(4q^2 + 4q + 1) + 2$.

Case 2: $n = 4q + 3$. Then $n^2 + 2 = 16q^2 + 24q + 11 = 4(4q^2 + 6q + 2) + 3$.

In all four cases, the remainder of n divided by 4 is not zero, so n isn't divisible by 4.

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Use proof by contrapositive to prove the following claim, using your best mathematical style and working directly from the definitions of "odd" and "even." (You may assume that odd and even are opposites.)

For all real numbers x and y, if $3x + y^2 + 2$ is odd, then x is even or y is even.

You must begin by explicitly stating the contrapositive of the claim:

Solution: Let's prove the contrapositive. That is, for all real numbers x and y, if x is odd and y is odd, then $3x + y^2 + 2$ is even.

Let x and y be real numbers. Suppose that x and y are both odd. Then there are integers p and q such that $x = 2p + 1$ and $y = 2q + 1$.

Then

$$\begin{aligned} 3x + y^2 + 2 &= 3(2p+1) + (2q+1)^2 + 2 \\ &= (6p+3) + (4q^2 + 4q + 1) + 2 \\ &= 6p + 4q^2 + 4q + 6 \\ &= 2(3p + 2q^2 + 2q + 3) \end{aligned}$$

Let $t = 3p + 2q^2 + 2q + 3$. The above shows that $3x + y^2 + 2 = 2t$. Furthermore t must be an integer because p and q are integers. So $3x + y^2 + 2$ must be even.

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Prove the following claim, working directly from the definitions of "remainder" and "divides", and using your best mathematical style.

For all real numbers k, m, n and r, if $r = \text{remainder}(m, n)$, $k \mid m$, and $k \mid n$, then $k \mid r$.

Solution: Let k, m, n and r be real numbers. Suppose that $r = \text{remainder}(m, n)$, $k \mid m$, and $k \mid n$.

By the definition of remainder, $m = nq + r$, where q is some integer. (Also r has to be between 0 and n, but that's not required here.) So $r = m - nq$.

By the definition of divides, $m = ks$ and $n = kt$, for some integers s and t. Substituting these into the previous equation, we get

$$r = m - nq = ks - ktq = k(s - tq)$$

$s - tq$ is an integer because s, t, and q are integers. So r is the product of k and an integer, which means that $k \mid r$.