CS 173, Fall 2015
Examlet 12, Part A

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Discussion: Thursday 2 3 4 5 Friday 9 10 11 12 1 2

(a) (9 points) A triomino is a triangular tile with a number on each edge. In our set of triominos, the numbers range from 0 to 5. So possible tiles include 5-3-4, 0-4-4, and 3-3-3. The back of each tile has a pretty pattern, so tiles can't be turned over. However, notice that a tile is the same if you rotate it, e.g. 5-3-4 is the same tile as 3-4-5. How many distinct tiles are in our set?

Solution:

All three sides have the same number: 6 tiles.

Two sides have the same number: 6 choices for the duplicated number, five choices for the single number. So 30 different tiles.

All three sides have different numbers: There are $6 \cdot 5 \cdot 4 = 120$ ways to pick an ordered list of three distinct numbers. But each ordered list is the rotated version of two other lists. So we need to divide by three to eliminate the overcounting, giving us 40 tiles.

Total number of tiles is 76.

(b) (6 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

> There is a bug b, such that for every plant p, if b pollinates p and p is showy, then p is poisonous.

Solution: For every bug b, there is a plant p, such that b pollinates p and p is showy, but p is not poisonous.

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(a) (9 points) In the polynomial $(x + y + z)^{30}$, what is the coefficient of the $x^{15}y^6z^9$ term. Briefly justify your answer.

Solution: The coefficient is the number of ways to fill 30 positions with 15 x's, 6 y's, and 9 z's.

There are $\binom{30}{15}$ ways to choose a set of positions to contain x's. Then there are $\binom{15}{6}$ ways to choose 6 of the remaining positions to contain y's. So the total number of choices is

$$\binom{30}{15} \cdot \binom{15}{6} = \frac{30!}{15! \cdot 6! \cdot 9!}.$$

(b) (6 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

For every cat c, if c is not fierce or c wears a collar, then c is a pet.

Solution: There exists a cat c that is either not fierce or wears a collar and is not a pet.

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(a) (9 points) Use proof by contradiction to prove that, for all real numbers a and b, if a is rational and b is irrational, then $a + b$ is irrational. (You must use the definition of "rational." You may not use facts about adding/subtracting rational numbers.)

Solution: Suppose not. That is, suppose that there are real numbers a and b such that a is rational, b is irrational, and $a + b$ is rational.

By the definition of rational, this means that $a = \frac{p}{q}$ and $a + b = \frac{m}{n}$ for some integers p, q, m, and n (q and n non-zero).

Then $b = (a + b) - a = \frac{m}{n} - \frac{p}{q} = \frac{mq-pn}{nq}$.

Notice that $mq - pn$ and nq are integers, since m, q, p, and n are integers. So b is the ratio of two integers and therefore rational. But this contradicts our assumption that b was irrational.

Since our original assumption led to a contradiction, it must be the case that for all real numbers a and b, if a is rational and b is irrational, then $a + b$ is irrational.

(b) (6 points) In the game Tic-tac-toe is played on a 3x3 grid and a move consists of the first player putting an X into one of the squares, or the second player putting an O into one of the squares. The board cannot be rotated, e.g. an X in the upper right corner is different from an X in the lower left corner. How many different board configurations are possible after four moves (i.e. two moves by each player)?

Solution: You need to pick 2 of the 9 squares to contain the X's, and then 2 of the remaining 7 squares to contain the O's. So the total number of choices is $\binom{9}{2}\binom{7}{2}$.

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(a) (9 points) Use proof by contradiction to show that $\sqrt{21} \leq \sqrt{7} + \sqrt{5}$.

Solution: Suppose not. That is, suppose that $\sqrt{21} > \sqrt{7} + \sqrt{5}$.

Then $21 > (\sqrt{7} + \sqrt{5}) = 7 + 2\sqrt{7}\sqrt{5} + 5$. So $9 > 2\sqrt{7}\sqrt{5}$.

Squaring both sides again, we get $81 > 4 \cdot 7 \cdot 5 = 140$. But $81 > 140$ is clearly false. So our original assumption must have been wrong and therefore $\sqrt{21} \leq \sqrt{7} + \sqrt{5}$.

(b) (6 points) Someone left the following equation on the hallway whiteboard. Is it correct? Explain why or why not. (Be clear but a formal proof is not required.)

$$\sum_{k=0}^{50} \binom{101}{2k} = \frac{1}{2} \sum_{k=0}^{101} \binom{101}{k}$$

Solution: The shorter summation $P = \sum_{k=0}^{50} \binom{101}{2k}$ contains the even terms from the longer summation $Q = \sum_{k=0}^{101} \binom{101}{k}$. Because the longer summation has even length, there are the same number of odd terms. Let $R = \sum_{k=0}^{50} \binom{101}{2k+1}$ be the summation containing the odd terms.

Notice that $\binom{101}{k} = \binom{101}{101-k}$. Because 101 is odd, k is even if and only if $101 - k$ is odd. So each even term can be paired with a corresponding odd term that's equal to it. So P must equal R. But $Q = P + R$, so $P = \frac{1}{2}Q$.

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(a) (9 points) Use proof by contradiction to show that $\sqrt{2}+\sqrt{3} \leq 4$.

Solution: Suppose not. That is, suppose that $\sqrt{2}+\sqrt{3} > 4$.

Then $(\sqrt{2}+\sqrt{3})^2 > 16$. (All the numbers involved are positive.) So $2+2\sqrt{2}\sqrt{3}+3 > 16$. So $2\sqrt{2}\sqrt{3} > 11$.

Squaring both sides again, we get $4 \cdot 2 \cdot 3 > 121$. That is $24 > 121$. But this last equation is obviously false. So our original assumption must have been wrong and therefore $\sqrt{2}+\sqrt{3} \leq 4$.

(b) (6 points) Suppose that w, x, y, and z are **positive** integers. How many solutions are there for the equation $w+x+y+z = 120$? Briefly explain or show work.

Solution: Since w, x, y, and z cannot be zero, we can rewrite our equation as $w'+x'+y'+z' = 116$, where $w' = w-1$, $x' = x-1$, $y' = y-1$, and $z' = z-1$. We can then view this as a problem of combinations with repetition, where we have four types (w, x, y, and z) and 116 objects. So the number of solutions is

$$\binom{119}{3} = \binom{119}{116}$$

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(a) (9 points) Suppose we know that $\sqrt{6}$ is irrational. Use proof by contradiction to show that $\sqrt{2}+\sqrt{3}$ is irrational. (You must use the definition of "rational." You may not use facts about adding/subtracting rational numbers.)

Solution: Suppose not. That is, suppose that $\sqrt{2}+\sqrt{3}$ is rational. Then there are integers p and q (q non-zero) such that $\sqrt{2}+\sqrt{3}=\frac{p}{q}$.

Squaring both sides of this equation gives $2+2\sqrt{6}+3=\frac{p^2}{q^2}$. So $2\sqrt{6}=\frac{p^2}{q^2}-5=\frac{p^2-5q^2}{q^2}$. So $\sqrt{6}=\frac{p^2}{q^2}-5=\frac{p^2-5q^2}{2q^2}$. But notice that p^2-5q^2 and $2q^2$ are both integers since p and q are integers. So this means that $\sqrt{6}$ is the ratio of two integers and therefore rational. But we know that $\sqrt{6}$ is not rational.

Since our original assumption led to a contradiction, $\sqrt{2}+\sqrt{3}$ must be irrational.

(b) (6 points) The bus to the Hackathon has 40 seats. In how many ways can I divide these seats among the CS, ECE, Math, and Physics departments?

Solution: This is a combinations with repetition problem, with four kinds (the departments) and 40 objects (i.e. the seats). So the total number of choices is $\binom{43}{3}$