CS 173, Fall 2015
Examlet 10, Part A

NETID:

FIRST: **LAST:**

Discussion: Thursday 2 3 4 5 Friday 9 10 11 12 1 2

(15 points) Use (strong) induction to prove the following claim:

Claim: For any sets A_1, A_2, …, A_n, $|A_1 \cup A_2 \cup \ldots \cup A_n| \leq |A_1| + |A_2| + \ldots + |A_n|$

Hint: remember the "Inclusion-Exclusion" formula for computing $|A \cup B|$ in terms of $|A|$, $|B|$, $|A \cap B|$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$ the claim reduces to $|A_1| \leq |A_1|$, which is clearly true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $|A_1 \cup A_2 \cup \ldots \cup A_n| \leq |A_1| + |A_2| + \ldots + |A_n|$, for any sets A_1, A_2, …, A_n, where $n = 1, 2, \ldots, k$.

Inductive Step: Let A_1, A_2, …, A_{k+1} be sets. Let $S = A_1 \cup A_2 \cup \ldots \cup A_k$.

We know that $|S \cup A_{k+1}| = |S| + |A_{k+1}| - |S \cap A_{k+1}|$ by the Inclusion-Exclusion formula. So $|S \cup A_{k+1}| \leq |S| + |A_{k+1}|$ because $|S \cap A_{k+1}|$ cannot be negative.

By the inductive hypothesis $|S| = |A_1 \cup A_2 \cup \ldots \cup A_k| \leq |A_1| + |A_2| + \ldots + |A_k|$.

So $|A_1 \cup A_2 \cup \ldots \cup A_{k+1}| = |S \cup A_{k+1}| \leq |S| + |A_{k+1}| \leq (|A_1| + |A_2| + \ldots + |A_k|) + |A_{k+1}|$.

So $|A_1 \cup A_2 \cup \ldots \cup A_{k+1}| \leq |A_1| + |A_2| + \ldots + |A_{k+1}|$, which is what we needed to prove.

CS 173, Fall 2015
Examlet 10, Part A

NETID:

FIRST: **LAST:**

Discussion: Thursday 2 3 4 5 Friday 9 10 11 12 1 2

(15 points) Use (strong) induction to prove the following claim:

Claim: For any natural number n and any real number x, where $0 < x < 1$, $(1-x)^n \geq 1-nx$.

Let x be a real number, where $0 < x < 1$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 0$, $(1-x)^n = (1-x)^0 = 1$ and $1 - nx = 1 + 0 = 1$. So $(1-x)^n \geq 1 - nx$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $(1-x)^n \geq 1 - nx$ for any natural number $n \leq k$, where k is a natural number.

Inductive Step: By the inductive hypothesis $(1-x)^k \geq 1-kx$. Notice that $(1-x)$ is positive since $0 < x < 1$. So $(1-x)^{k+1} \geq (1-x)(1-kx)$.

But $(1-x)(1-kx) = 1 - x - kx + kx^2 = 1 - (1+k)x + kx^2$.

And $1 - (1+k)x + kx^2 \geq 1 - (1+k)x$ because kx^2 is non-negative.

So $(1-x)^{k+1} \geq (1-x)(1-kx) \geq 1-(1+k)x$, and therefore $(1-x)^{k+1} \geq 1-(1+k)x$, which is what we needed to show.

CS 173, Fall 2015
Examlet 10, Part A

NETID:

FIRST: **LAST:**

Discussion: Thursday 2 3 4 5 Friday 9 10 11 12 1 2

(15 points) Use (strong) induction to prove the following claim:

Claim: $\sum_{p=2}^{n} \frac{1}{p^2} \leq \frac{3}{4} - \frac{1}{n}$ for all integers $n \geq 2$

Solution: Proof by induction on n.

Base Case(s): At $n = 2$, $\sum_{p=2}^{n} \frac{1}{p^2} = \frac{1}{4} \leq \frac{3}{4} - \frac{1}{2}$. So the claim holds at $n = 2$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\sum_{p=2}^{n} \frac{1}{p^2} \leq \frac{3}{4} - \frac{1}{n}$ for $n = 2, 3, \ldots, k$

Inductive Step: Notice that $\frac{1}{(k+1)^2} \leq \frac{1}{k(k+1)} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k} - \frac{1}{(k+1)}$.

So $-frac1k + \frac{1}{(k+1)^2} \leq -\frac{1}{(k+1)}$.

So $\frac{3}{4} - frac1k + \frac{1}{(k+1)^2} \leq \frac{3}{4} - \frac{1}{(k+1)}$.

By the inductive hypothesis, we know that $\sum_{p=2}^{k} \frac{1}{p^2} \leq \frac{3}{4} - \frac{1}{k}$. Using this fact and the above work, we can compute:

$\sum_{p=2}^{k+1} \frac{1}{p^2} = \sum_{p=2}^{k} \frac{1}{p^2} + \frac{1}{(k+1)^2} \leq (\frac{3}{4} - \frac{1}{k}) + \frac{1}{(k+1)^2} \leq \frac{3}{4} - \frac{1}{(k+1)}$

So $\sum_{p=2}^{k+1} \frac{1}{p^2} \leq \frac{3}{4} - \frac{1}{k+1}$, which is what we needed to show.

CS 173, Fall 2015
Examlet 10, Part A

NETID:

FIRST: | **LAST:**

Discussion: **Thursday** **2** **3** **4** **5** **Friday** **9** **10** **11** **12** **1** **2**

(15 points) Use (strong) induction to prove the following claim:

Claim: For any positive integer n, $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \geq \sqrt{n}$

You may use the fact that $\sqrt{n+1} \geq \sqrt{n}$ for any natural number n.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} = 1$ Also $\sqrt{n} = 1$. So $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \geq \sqrt{n}$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \geq \sqrt{n}$ for $n = 1, 2, \ldots, k$, for some integer $k \geq 1$.

Inductive Step: $\sum_{p=1}^{k} \frac{1}{\sqrt{p}} \geq \sqrt{k}$ by the inductive hypothesis.

So

$$\sum_{p=1}^{k+1} \frac{1}{\sqrt{p}} = \frac{1}{\sqrt{k+1}} + \sum_{p=1}^{k} \frac{1}{\sqrt{p}} \geq \frac{1}{\sqrt{k+1}} + \sqrt{k} = \frac{1 + \sqrt{k}\sqrt{k+1}}{\sqrt{k+1}} \geq \frac{1 + \sqrt{k}\sqrt{k}}{\sqrt{k+1}} = \frac{1+k}{\sqrt{k+1}} = \sqrt{k+1}$$

So $\sum_{p=1}^{k+1} \frac{1}{\sqrt{p}} \geq \sqrt{k+1}$, which is what we needed to show.

CS 173, Fall 2015
Examlet 10, Part A

NETID:

FIRST: **LAST:**

Discussion: Thursday 2 3 4 5 Friday 9 10 11 12 1 2

(15 points) Use (strong) induction to prove the following claim:

Claim: $\displaystyle\sum_{k=n+1}^{2n} \frac{1}{k} \ \geq \ \frac{7}{12}$, for any integer $n \geq 2$.

Hint: recall that if $x \leq y$, then $\frac{1}{y} \leq \frac{1}{x}$

Solution:

Proof by induction on n.

Base Case(s): At $n = 2$, $\displaystyle\sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\displaystyle\sum_{k=n+1}^{2n} \frac{1}{k} \ \geq \ \frac{7}{12}$, for $n = 2, 3, \ldots, p$.

Inductive Step: Substituing $n = p+1$ into the summation and then using the inductive hypothesis, we get

$$\sum_{k=p+2}^{2p+2} \frac{1}{k} \ = \ (\sum_{k=p+1}^{2p} \frac{1}{k}) + (\frac{1}{2p+1} + \frac{1}{2p+2} - \frac{1}{p+1}) \geq \frac{7}{12} + (\frac{1}{2p+1} + \frac{1}{2p+2} - \frac{1}{p+1})$$

Now, notice that $\frac{1}{2p+1} \geq \frac{1}{2}\frac{1}{p+1}$ and $\frac{1}{2p+2} = \frac{1}{2}\frac{1}{p+1}$. So $\frac{1}{2p+1} + \frac{1}{2p+2} \geq \frac{1}{p+1}$. Therefore $\frac{1}{2p+1} + \frac{1}{2p+2} - \frac{1}{p+1} \geq 0$.

Combining the results of the previous two paragraphs, we get

$$\sum_{k=p+2}^{2p+2} \frac{1}{k} \geq \frac{7}{12} + (\frac{1}{2p+1} + \frac{1}{2p+2} - \frac{1}{p+1}) \geq \frac{7}{12}$$

This is what we needed to show.

CS 173, Fall 2015
Examlet 10, Part A

NETID:

FIRST: **LAST:**

Discussion: Thursday 2 3 4 5 Friday 9 10 11 12 1 2

(15 points) Let function $f : \mathbb{Z}^+ \to \mathbb{N}$ be defined by

$f(1) = 0$

$f(n) = 1 + f(\lfloor n/2 \rfloor)$, for $n \geq 2$,

Use (strong) induction on n to prove that $f(n) \leq \log_2 n$ for any positive integer n. You cannot assume that n is a power of 2. However, you can assume that the log function is increasing (if $x \leq y$ then $\log x \leq \log y$) and that $\lfloor x \rfloor \leq x$.

Solution:

Proof by induction on n.

Base Case(s):

$f(1) = 0$ and $\log_2 1 = 0$ So $f(1) \leq \log_2 1$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $f(n) \leq \log_2 n$ for $n = 1, \ldots, k-1$.

Inductive Step:

We can assume that $k \geq 2$ (since we did $n = 1$ for the base case). So $\lfloor k/2 \rfloor$ must be at least 1 and less than k. Therefore, by the inductive hypothesis, $f(\lfloor k/2 \rfloor) \leq \log_2(\lfloor k/2 \rfloor)$.

We know that $f(k) = 1 + f(\lfloor k/2 \rfloor)$, by the definition of f. Substituting the result of the previous paragraph, we get that $f(k) \leq 1 + \log_2(\lfloor k/2 \rfloor)$.

$\lfloor k/2 \rfloor \leq k/2$. So $\log_2(\lfloor k/2 \rfloor) \leq \log_2(k/2) = (\log_2 k) + (\log_2 1/2) = (\log_2 k) - 1$.

Since $f(k) \leq 1 + \log_2(\lfloor k/2 \rfloor)$ and $\log_2(\lfloor k/2 \rfloor) \leq (\log_2 k) - 1$, $f(k) \leq 1 + (\log_2 k) - 1 = (\log_2 k)$. This is what we needed to show.