1. Consider the following recursive definition:

\[
\begin{align*}
f(1) &= 3 \\
f(n) &= 3f(n-1) + n^3 \text{ for } n \geq 2
\end{align*}
\]

Express \( f(n) \) in terms of \( f(n-3) \) (where \( n \geq 4 \)). Show your work and simplify your answer.

**Solution:**

Let \( n \geq 4 \).

\[
\begin{align*}
f(n) &= 3f(n-1) + n^3 \\
&= 3(3f(n-2) + (n-1)^3) + n^3 \\
&= 9f(n-2) + 3(n-1)^3 + n^3 \\
&= 9(3f(n-3) + (n-2)^3) + 3(n-1)^3 + n^3 \\
&= 27f(n-3) + 9(n-2)^3 + 3(n-1)^3 + n^3
\end{align*}
\]
2. Recall that the Fibonacci series is defined as:
   \[ F(0) = 0; \ F(1) = 1; \ F(i) = F(i-1) + F(i-2), \text{ for every } i > 1. \]

   Prove that
   \[ F(0)^2 + F(1)^2 + \ldots + F(n)^2 = F(n)F(n+1) \]
   for every \( n \in \mathbb{N} \), by induction on \( n \).

   **Solution:**

   **Proof:**

   Let us prove by induction on \( n \) that
   \[ P(n) : F(0)^2 + F(1)^2 + \ldots + F(n)^2 = F(n)F(n+1) \]
   holds, for every \( n \in \mathbb{N} \).

   **Base case:** \( n = 0 \)
   \[ F(0)^2 + F(1)^2 + \ldots + F(0)^2 = F(0)^2 = 0, \text{ and } F(n)F(n+1) = 0 \cdot 1 = 0. \]
   Hence claim holds.

   **Induction step:**
   Let \( n \in \mathbb{N} \) and \( n > 0 \).
   Let us assume the induction hypothesis:
   \[ \text{for every } j \in \mathbb{N}, \ j < n, \ F(0)^2 + F(1)^2 + \ldots + F(j)^2 = F(j)F(j+1) \]
   Now, since \( n - 1 \in \mathbb{N} \) (since \( n > 0 \)) and \( n - 1 < n \), by the induction hypothesis,
   \[ F(0)^2 + F(1)^2 + \ldots + F(n-1)^2 = F(n-1)F(n) \]
   Hence
   \[
   \begin{align*}
   & F(0)^2 + F(1)^2 + \ldots + F(n-1)^2 + F(n)^2 \\
   = & F(n-1)F(n) + F(n)^2 \quad \text{(by ind. hypothesis)} \\
   = & F(n)(F(n-1) + F(n)) \\
   = & F(n)F(n+1) \quad (\text{since } F(n+1) = F(n) + F(n-1))
   \end{align*}
   \]
   which proves the claim.

   Hence, we have proved the claim by induction. \( \quad \text{QED.} \)
3. Prove that any collection of eight distinct integers contains distinct integers $x$ and $y$ such that $x - y$ is a multiple of 7.

Proof:

Let us consider eight distinct integers.
Consider the remainder when you divide these integers by 7 (i.e., for each integer, consider which congruence class modulo 7 they belong to).
Since there are only 7 congruence classes, and there are 8 integers, by the pigeon-hole principle, there must be two (distinct) integers $x$ and $y$ among them such that $x \equiv y \mod 7$.
Now, by definition, $7|(x - y)$.
Hence there must be two distinct integers whose difference is a multiple of 7.
4. Let $R$ be defined recursively as follows:
\[
R(1) = 2 \\
R(2) = 8 \\
R(n) = 2R(n-1) + 3R(n-2) + 4
\]
Prove that $R(n) = 3^n - 1$ for every $n > 0$, by induction on $n$.

Solution:

Proof:
Let us prove by induction on $n$ that

\[
P(n) : R(n) = 3^n - 1
\]
holds, for every $n \in \mathbb{N}$, $n > 0$.

**Base cases:**
Case: $n = 1$

$R(1) = 2$ and $3^1 - 1 = 2$. Hence claim holds when $n = 1$.

Case: $n = 2$

$R(2) = 8$ and $3^2 - 1 = 8$. Hence claim holds when $n = 2$.

**Induction step:**
Let $n > 2$ be an arbitrary natural number.

Let us assume the *induction hypothesis*:

For any $j \in \mathbb{N}$, $j > 0$ and $j < n$, $R(j) = 3^j - 1$.

Now, since $n > 1$,

\[
R(n) &= 2R(n-1) + 3R(n-2) + 4 \\
&= 2 \cdot (3^{n-1} - 1) + 3 \cdot (3^{n-2} - 1) + 4 \quad (by \ ind \ hypo \ since \ n-1 > 0, n-2 > 0, \ and \ n-1, n-2 < n) \\
&= 2 \cdot 3^{n-1} + 3 \cdot 3^{n-2} - 2 - 3 + 4 \\
&= 6 \cdot 3^{n-2} + 3 \cdot 3^{n-2} - 1 \\
&= 9 \cdot 3^{n-2} - 1 \\
&= 3^n - 1
\]

which proves the claim.

We have hence shown, by induction on $n$, that $R(n) = 3^n - 1$, for every $n > 0$. QED