CS 173, Fall 2014, Examlet 4 SOLUTIONS, Part I

Discussion Sec. (circle yours):

FRI-11 **FRI-12** **FRI-1** **FRI-2** **FRI-3** **FRI-4**

LASTNAME, FIRSTNAME (in CAP letters): **NETID:**

Points: Q1: 5, Q2: 15

1. Consider the following recursive definition:

$$\begin{aligned} f(1) &= 3 \\ f(n) &= 3f(n-1) + n^3 \text{ for } n \geq 2 \end{aligned}$$

Express $f(n)$ in terms of $f(n-3)$ (where $n \geq 4$). Show your work and simplify your answer.

Solution:

Let $n \geq 4$.

$$\begin{aligned} f(n) &= 3f(n-1) + n^3 \\ &= 3(3f(n-2) + (n-1)^3) + n^3 \\ &= 9f(n-2) + 3(n-1)^3 + n^3 \\ &= 9(3f(n-3) + (n-2)^3) + 3(n-1)^3 + n^3 \\ &= 27f(n-3) + 9(n-2)^3 + 3(n-1)^3 + n^3 \end{aligned}$$

⟨⟨SEE REVERSE SIDE FOR PROBLEM 2⟩⟩

2. Recall that the Fibonacci series is defined as:
$F(0) = 0$; $F(1) = 1$; $F(i) = F(i-1) + F(i-2)$, for every $i > 1$.

Prove that

$$F(0)^2 + F(1)^2 + \ldots + F(n)^2 = F(n)F(n+1)$$

for every $n \in \mathbb{N}$, by induction on n.

Solution:

Proof:

Let us prove by induction on n that

$$P(n) : F(0)^2 + F(1)^2 + \ldots + F(n)^2 = F(n)F(n+1)$$

holds, for every $n \in \mathbb{N}$.

Base case: $n = 0$
$F(0)^2 + F(1)^2 + \ldots + F(n)^2 = F(0)^2 = 0$, and $F(n)F(n+1) = 0 \cdot 1 = 0$. Hence claim holds.

Induction step:
Let $n \in \mathbb{N}$ and $n > 0$.

Let us assume the *induction hypothesis*:
for every $j \in \mathbb{N}$, $j < n$, $F(0)^2 + F(1)^2 + \ldots + F(j)^2 = F(j)F(j+1)$

Now, since $n - 1 \in \mathbb{N}$ (since $n > 0$) and $n - 1 < n$, by the induction hypothesis,
$F(0)^2 + F(1)^2 + \ldots + F(n-1)^2 = F(n-1)F(n)$

Hence

$$\begin{aligned}
&F(0)^2 + F(1)^2 + \ldots + F(n-1)^2 + F(n)^2 \\
&= F(n-1)F(n) + F(n)^2 \qquad \textit{(by ind. hypothesis)} \\
&= F(n)(F(n-1) + F(n)) \\
&= F(n)F(n+1) \qquad \textit{(since } F(n+1) = F(n) + F(n-1)\textit{)}
\end{aligned}$$

which proves the claim.

Hence, we have proved the claim by induction. QED.

CS 173, Fall 2014, Examlet 4, Part II

Discussion Sec. (circle yours):

FRI-11 **FRI-12** **FRI-1** **FRI-2** **FRI-3** **FRI-4**

LASTNAME, FIRSTNAME (in CAP letters): **NETID:**

Points: Q3: 10; Q4: 15

3. Prove that any collection of eight distinct integers contains distinct integers x and y such that $x - y$ is a multiple of 7.

   Proof:

   Let us consider eight distinct integers.
   Consider the remainder when you divide these integers by 7 (i.e., for each integer, consider which congruence class modulo 7 they belong to).
   Since there are only 7 congruence classes, and there are 8 integers, by the pigeon-hole principle, there must be two (distinct) integers x and y among them such that $x \equiv y \ mod\ 7$.

   Now, by definition, $7|(x - y)$.

   Hence there must be two distinct integers whose difference is a multiple of 7.

⟨⟨SEE REVERSE SIDE FOR PROBLEM 4⟩⟩

4. Let R be defined recursively as follows:

$R(1) = 2$
$R(2) = 8$
$R(n) = 2R(n-1) + 3R(n-2) + 4$

Prove that $R(n) = 3^n - 1$ for every $n > 0$, by induction on n.

Solution:

Proof:

Let us prove by induction on n that

$$P(n) : R(n) = 3^n - 1$$

holds, for every $n \in \mathbb{N}$, $n > 0$.

Base cases:

Case: $n = 1$
$R(1) = 2$ and $3^1 - 1 = 2$. Hence claim holds when $n = 1$.

Case: $n = 2$
$R(2) = 8$ and $3^2 - 1 = 8$. Hence claim holds when $n = 2$.

Induction step:
Let $n > 2$ be an arbitrary natural number.

Let us assume the *induction hypothesis*:
For any $j \in \mathbb{N}$, $j > 0$ and $j < n$, $R(j) = 3^j - 1$.

Now, since $n > 1$,

$$\begin{aligned}
R(n) &= 2R(n-1) + 3R(n-2) + 4 \\
&= 2 \cdot (3^{n-1} - 1) + 3 \cdot (3^{n-2} - 1) + 4 \quad \textit{(by ind hypo since } n\text{–}1 > 0, n\text{–}2 > 0, \textit{ and } n\text{–}1, n\text{–}2 < n\textit{)} \\
&= 2 \cdot 3^{n-1} + 3 \cdot 3^{n-2} - 2 - 3 + 4 \\
&= 6 \cdot 3^{n-2} + 3 \cdot 3^{n-2} - 1 \\
&= 9 \cdot 3^{n-2} - 1 \\
&= 3^n - 1
\end{aligned}$$

which proves the claim.

We have hence shown, by induction on n, that $R(n) = 3^n - 1$, for every $n > 0$. QED