CS 173, Fall 2014
Examlet 9 Part A

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Discussion: **Thursday** **2** **3** **4** **5** **Friday** **9** **10** **11** **12** **1** **2**

(18 points) D-trees are binary trees whose nodes are labelled with strings, such that

- Each leaf node has label `left`, `right`, or `back`
- If a node has one child, it has label α`x` where where α is the child's label. E.g. if the child has label `left` then the parent has `leftx`.
- If a node has two children, it contains $\alpha\beta$ where α and β are the child labels. E.g. if the children have labels `right` and `back`, then the parent has label `rightback`.

Let S(n) be the length of the label on node n. Let L(n) be the number of leaves in the subtree rooted at n. Use (strong) induction to prove that $S(n) \geq 4L(n)$.

The induction variable is named **h** and it is the **height** of/in the tree.

Base case(s): $h = 0$. The tree consists of a single leaf node, so $L(n) = 1$. The node has label `left`, `right`, or `back`, so $S(n) \geq 4$. So $S(n) \geq 4L(n)$.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $S(n) \geq 4L(n)$ if n is the root node of any D-tree of height $< k$ (where $k \geq 1$).

Rest of the inductive step:

Suppose that T is a D-tree of height k. There are two cases:

Case 1: The root n of T has a single child node p. By the inductive hypothesis $S(p) \geq 4L(p)$. $L(n) = L(p)$. And $S(n) = S(p) + 1$. So $S(n) \geq 4L(n)$.

Case 2: The root n of T has two children p and q. By the inductive hypothesis $S(p) \geq 4L(p)$ and $S(q) \geq 4L(q)$.

Notice that $L(n) = L(p) + L(q)$. And $S(n) = S(p) + S(q)$.

So $S(n) = S(p) + S(q) \geq 4L(p) + 4L(q) = 4L(n)$.

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(18 points) A Raven tree is a binary tree whose nodes contain integers such that

- Every leaf node contains 7, 9, or 12.
- A node with one child contains the same number as its child.
- A node with two children contains the value $xy - y$, where x and y are the values in its children.

Use (strong) induction to prove that the value in the root of a Raven tree is always ≥ 7

The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): $h = 0$. Raven trees of height zero have a single node, which is both the root and a leaf. So it contains 7, 9, or 12, all of which are ≥ 7.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that the value in the root node of a Raven tree is always ≥ 7, for all trees of height $h = 0, 1, \ldots, k-1$ (k an integer ≥ 1).

Inductive Step: Let T be a Raven tree of height k ($k \geq 1$). There are two cases:

Case 1: T consists of a root with a single subtree under it. Call the subtree T_1. By the inductive hypothesis, the root of T_1 contains a value ≥ 7. By the definition of Raven trees, the root of T contains the same value, which is therefore also ≥ 7.

Case 2: T consists of a root with two subtrees T_1 and T_2 under it. Suppose the roots of the subtrees contain values x and y. By the inductive hypothesis $x \geq 7$ and $y \geq 7$.

The root of T then has value $xy - y$ by the definition of Raven trees. Since $x \geq 7$, $x - 1 \geq 6 \geq 1$. So $xy - y = (x-1)y \geq y \geq 7$.

In both cases the root of T contains a value ≥ 7, which is what we needed to show.

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(18 points) Let's define a Diagonal tree to be a binary tree containing 2D points such that:

- Each leaf node contains $(-1, -1)$, $(2, 8)$, or $(3, 11)$.
- An internal node with one child labelled (a, b) has label $(a + 2, b + 6)$.
- An internal node with two childen labelled (x, y) and (a, b) has label $(x + a, y + b - 2)$.

Use (strong) induction to prove that the root node of any Diagonal tree has a label on the line $y = 3x+2$.

The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): $h = 0$ for the base case, i.e. each tree consists of a single node that is both the root and a leaf. So the node must have label $(-1, -1)$, $(2, 8)$, or $(3, 11)$. All three of these points lie on the line $y = 3x + 2$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Any Diagonal tree of height less than k ($k \geq 1$) has a root node whose label is a point on the line $y = 3x + 2$.

Inductive Step:

Let T be a Diagonal tree of height k. There are two cases:

Case 1: The root has a single child. Suppose the child has label (a, b). Then the root of T has label $(a+2, b+6)$. By the inductive hypothesis, (a, b) must lie on the line $y = 3x+2$. That is $b = 3x+2$. Then $b + 6 = (3x + 2) + 6 = 3x + 6 + 2 = 3(x + 2) + 2$. So the point $(a + 2, b + 6)$ lies on the line $y = 3x + 2$, which is what we needed to show.

Case 2: The root has two children. Suppose that their labels are (p, q) and (a, b). Then the root of T has label $(p + a, q + b - 2)$. By the inductive hypothesis, (p, q) and (a, b) must lie on the line $y = 3x + 2$. That is $q = 3p + 2$ and $b = 3a + 2$.

Adding these two equations together, we get that $q + b = 3p + 2 + 3a + 2$. So $q + b - 2 = 3(p + a) + 2$. Therefore, the point $(p + a, q + b - 2)$ lies on the line $y = 3x + 2$, which is what we needed to show.

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(18 points) Here is a grammar G, with start symbol S and terminal symbols a and p.

$$S \rightarrow S\,S \mid p\,S\,p \mid p\,p \mid a\,a$$

Use (strong) induction to prove that any tree matching (aka generated by) grammar G has an even number of nodes with label p. Use $P(T)$ as shorthand for the number of p's in a tree T.

The induction variable is named **h** and it is the **height** of/in the tree.

Base Case(s): The shortest trees matching grammar G have height $h = 1$. There are two such trees, which look like

```
    S                S
  /   \            /   \
 p     p          a     a
```

Both of these contain an even number of nodes with label p.

Inductive Hypothesis— [Be specific, don't just refer to "the claim"]:

Suppose that all trees T matching grammar G with heights $h = 1, 2, \ldots, k-1$ have $P(T)$ even, for some integer $k \geq 2$.

Inductive Step: Let T be a tree of height k matching grammar G, where $k \geq 2$. There are two cases:

Case 1: T consists of a root with label S plus two child subtrees T_1 and T_2. By the inductive hypothesis $P(T_1)$ and $P(T_2)$ are both even. But $P(T) = P(T_1) + P(T_2)$. So $P(T)$ is also even.

Case 2: T consists of a root with label S plus three children. The left and right children are single nodes containing label p. The center child is a subtree T_1. By the inductive hypothesis, $P(T_1)$ is even. $P(T) = P(T_1) + 2$. So $P(T)$ is also even.

In both cases $P(T)$ is even, which is what we needed to show.