CS 173, Fall 2014
Examlet 3, Part A

NETID:

FIRST:	**LAST:**

Discussion: Thursday 2 3 4 5 Friday 9 10 11 12 1 2

$A = \{(a, b) : (a, b) \in \mathbb{R}^2, a = 3 - b^2\}$

$B = \{(x, y) : (x, y) \in \mathbb{R}^2, |x| \geq 1 \text{ or } |y| \geq 1\}$

Prove that A $\subseteq$ B. Hint: you may find proof by cases helpful.

Solution: Suppose that (a, b) is an element of A. Then, by the definition of A, $(a, b) \in \mathbb{R}^2$ and $a = 3 - b^2$.

Consider two cases, based on the magnitude of b:

Case 1: $|b| \geq 1$. Then (a, b) is an element of B. (Because it satisfies one of the two conditions in the OR.)

Case 2: $|b| < 1$. Then $b^2 < 1$. Then $a = 3 - b^2 > 3 - 1 = 2$. So $|a| \geq 1$, which means that (a, b) is an element of B.

So (a, b) is an element of B in both cases, which is what we needed to show.

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$A = \{(x, y) \in \mathbb{Z}^2 \mid 2xy + 6y - 5x - 15 \geq 0\}$

$B = \{(a, b) \in \mathbb{Z}^2 \mid a \geq 0\}$

$C = \{(p, q) \in \mathbb{Z}^2 \mid q \geq 0\}$

Prove that $(A \cap B) \subseteq C$.

Solution: Suppose that (x, y) is an element of $(A \cap B)$. This means that (x, y) is an element of A and (x, y) is an element of B. So $2xy + 6y - 5x - 15 \geq 0$ and $x \geq 0$, by the definitions of A and B.

Notice that $2xy + 6y - 5x - 15 = (x + 3)(2y - 5)$. So $(x + 3)(2y - 5) \geq 0$. We know that $x + 3$ is positive because $x \geq 0$. So we must have $(2y - 5) \geq 0$.

Now, if $(2y - 5) \geq 0$, then $2y \geq 5$. So $y \geq \frac{5}{2}$. So $y \geq 0$. This means that (x, y) is an element of C which is what we needed to show.

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$A = \{(x, x^2) \ : \ x \in \mathbb{R} \text{ and } x \geq 2\}$

$B = \{(3p - 2, p) \ : \ p \in \mathbb{R}\}$

$C = \{(x, y) \in \mathbb{R}^2 \ : \ x \geq 0\}$

Prove that $A \cap B \subseteq C$.

Solution 1: Let (x, y) be an element of $A \cap B$. Then (x, y) is an element of A and also an element of B. So, by the definition of A, $x \geq 2$ and $y = x^2$. By the definition of B, $x = 3y - 2$.

Since $y = x^2$ and $x \geq 2$, $y \geq 4$. Since $x = 3y - 2$, $x \geq 3 \cdot 3 - 2 = 10$. So $x \geq 0$. Therefore, (x, y) is an element of C, which is what we needed to show.

Comments: The above solution does work, but it's not very insightful: $A \cap B$ is the empty set! As you can probably guess, this isn't what I intended when I wrote the problem. Here's another approach to the problem, based on that observation.

Solution 2: Let (x, y) be an element of $A \cap B$. Then (x, y) is an element of A and also an element of B. So, by the definition of A, $x \geq 2$ and $y = x^2$. By the definition of B, $x = 3y - 2$.

Since $y = x^2$ and $x = 3y - 2$, $x = 3x^2 - 2$. So $2 = 3x^2 - x = x(3x - 1)$.

But also notice that since $x \geq 2$ and $y = x^2$, $y \geq 4$. So $x \geq 3 \cdot 4 - 2 = 10$. And therefore $3x - 1 \geq 29$.

Combining the two, we get $2 = 3x^2 - x \geq 10 \cdot 29 = 290$ But $2 \geq 290$ is impossible, so this supposed element (x, y) can't exist.

Therefore $A \cap B$ is the empty set, so it's vacuously true that $A \cap B \subseteq C$.

Comments: Or you could have used the quadratic formula, if you remember it better than I do.

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$A = \{(x, y) \in \mathbb{R}^2 \mid x = \lfloor 3y + 5 \rfloor \}$

$B = \{(p, q) \in \mathbb{Z}^2 \mid 2p + q \equiv 3 \pmod 7 \}$

Prove that $A \cap \mathbb{Z}^2 \subseteq B$.

Use the following definition of congruence mod k: if s, t, k are integers, k positive, then $s \equiv t \pmod k$ if and only if $s = t + nk$ for some integer n.

Solution: Let (x, y) be an element of $A \cap \mathbb{Z}^2$. Then (x, y) is an element of A and, also, both x and y are integers.

By the definition of Z, $x = \lfloor 3y + 5 \rfloor$. Since y is an integer, $3y + 5$ must also be an integer. So $\lfloor 3y + 5 \rfloor = 3y + 5$. Therefore, $x = 3y + 5$.

Now, consider $2x + y$.

$$2x + y = 2(3y + 5) + y = 7y + 10 = 7(y + 1) + 3$$

$y + 1$ is an integer, since y is an integer. So this means that $2x + y \equiv 3 \pmod 7$. Therefore, (x, y) is an element of B, which is what we needed to show.