1. **Graphs [8 points]**

Consider the following graphs $G_1$ and $G_2$.

$G_1$: 
- $F$ connected to $A$, $B$ and $C$
- $A$ connected to $B$ and $C$
- $C$ connected to $G$
- $G$ connected to $E$
- $E$ connected to $D$
- $D$ connected to $C$

$G_2$: 
- $1$ connected to $2$ and $3$
- $2$ connected to $4$
- $3$ connected to $5$
- $4$ connected to $6$
- $5$ connected to $7$

Is there an isomorphism between graphs $G_1$ and $G_2$? Prove your claim.

If you want to show that there is an isomorphism, show the appropriate function $f : V_1 \to V_2$. If you want to show there is no isomorphism, you need to prove that.

**Solution:** Yes, there is an isomorphism between $G_1$ and $G_2$. For example, here is one:

(a) $F \to 6$
(b) $B \to 7$
(c) $A \to 5$
(d) $G \to 4$
(e) $E \to 1$
(f) $C \to 2$
(g) $D \to 3$

2. **Chromatic Number [12 points]**

What is the chromatic number of the graph below? You need to carefully prove the chromatic number you give is correct by using 2-way bounding, establishing the lower bound and upper bound using separate arguments.

Moreover, in order to establish the upper bound, show a coloring of the graph by drawing a picture of the graph and labeling it with colors. Use a diagram drawing package to do this (see page on homework style under the Homework link on the course website for guidance).
Solution: 4 The chromatic number of \( G \), \( \chi(G) = 4 \). To establish this, we will establish 4 as an upper bound and a lower bound for the chromatic number.

(a) To establish an upper bound of 4, i.e., to show \( \chi(G) \leq 4 \) : we show that 4 colors are sufficient to color the graph. The picture below gives a 4-coloring of the graph using the colors \( \{R, G, B, Y\} \). Notice that no two adjacent nodes are colored using the same color.

(b) To show a lower bound of 4 for the chromatic number, i.e., to show \( \chi(G) \geq 4 \) :, notice that the graph contains \( W_5 \), the wheel graph with 5 nodes in a cycle \( (A, E, D, B, C) \) with a hub \( (F) \) in the middle, as a subgraph. Coloring an odd cycle requires clearly 3 colors at least (a 2-coloring is impossible). The middle hub, being adjacent to every node in the cycle, needs a new color. So \( W_5 \) requires at least 4 colors, and hence the entire graph requires at least 4 colors.
3. Graphs [8 points]

A set of people meet in a room, and over the day, people clink glasses with their friends. Prove that there always will be an even number of people who have done an odd number of clinks.

Give a clear modeling of this scenario as a graph, and a subsequent argument.

Solution: Let us model the scenario as a graph, where there is a node for every person in the room, and let us draw an edge between the nodes for person A and person B iff A and B have clinked glasses. Note that the number of clinks a person has done is the degree of the corresponding node in the graph. Hence, in order to prove that an even number of people have done an odd number of clinks, we have to hence prove that in this graph, there are an even number of nodes with odd degree. We will prove this by showing that in any simple graph with no self-loops, there are an even number of nodes with odd degree.

Let $G$ be any simple graph with no self-loops. Then by the handshaking theorem we know that:

\[ \sum_{v \in V} \deg(v) = 2|E| \]

where $V$ is the set of nodes (people) and $E$ is the set of edges (clinks).

Since the right-hand side is clearly even, the sum of degrees of all nodes is even. The sum of degrees of even-degree vertices is clearly even. Hence the sum of degrees of odd-degree vertices should be even as well. If the number of odd-degree vertices were odd, then the sum of degrees of odd-degree vertices would be odd, which is not the case. Hence the number of odd-degree vertices has to be even.

4. Proof: Induction [14 points] Use induction to prove that $\sum_{i=1}^{d} 2^i = 2^{d+1} - 2$

Solution: Proof by induction on $d$.

Base case: $d = 1$. $\sum_{i=1}^{1} 2^i = 2^1 = 2$. Also $2^{1+1} - 2 = 2^2 - 2 = 2$. Since they are equal, the claim is true at $d = 1$.

Induction hypothesis: Suppose that $\sum_{i=1}^{d} 2^i = 2^{d+1} - 2$ for $d = 1, \ldots, k - 1$. In particular, $\sum_{i=1}^{k-1} 2^i = 2^k - 2$.

Then, $\sum_{i=1}^{k} 2^i = (\sum_{i=1}^{k-1} 2^i) + 2^k = (2^k - 2) + 2^k = 2 \cdot 2^k - 2 = 2^{k+1} - 2$

Hence we have proved by induction on $d$ that for any positive integer $d$, $\sum_{i=1}^{d} 2^i = 2^{d+1} - 2$. 
