1. Functions I (12 points)

Recall that \( Z^+ \) is the set of positive integers. Let’s define the function \( h : (Z^+)^2 \to \mathbb{R} \) such that \( h(d, e) = 2^d + \frac{1}{e} \). Is \( h \) onto and/or one-to-one? Justify your answer.

By “justify,” we mean that you should provide an explanation that clearly shows your understanding of what this function does. However, you can describe its behavior informally, without the detail that a formal proof would require.

Solution:

The image of \( h \) consists of a little sequence of numbers near each positive power of two. The offset (from the power of 2) starts off with 1 and then continues with a set of increasingly small fractions. Because all the values in the image are clustered near powers of two, the image doesn’t cover anything like all of \( \mathbb{R} \), so it’s not onto. The offsets are small enough that the numbers associated with one power of two are separated from those associated with other powers of two, so this function is one-to-one.

2. Functions II (28 points)

Suppose that \( f : (0, \infty) \to (\frac{5}{4}, \infty) \) is defined by \( f(x) = \frac{5x^2 + 3}{4x^2} \).

(a) (14 points) Prove that \( f \) is onto.

(b) (14 points) Prove that \( f \) is one-to-one.

For both proofs, you may use well-known properties of the square root function (e.g. it’s one-to-one if restricted to positive inputs).

Solution:

(a) Let \( y \in (\frac{5}{4}, \infty) \). Consider \( x = \sqrt{\frac{3}{4y-5}} \). Since \( y > \frac{5}{4} \), \( 4y - 5 \) is always positive. So \( x \) is well defined (no dividing by zero). And, also \( x \) is positive, so \( x \) comes from the domain of \( f \).

Then \( x^2 = \frac{3}{4y-5} \).

So \( f(x) = \frac{5x^2 + 3}{4x^2} = \frac{5\cdot \frac{3}{4y-5} + 3}{4\cdot \frac{3}{4y-5}} \).

Multiplying by \( 4y - 5 \), we get \( f(x) = \frac{5\cdot \frac{3}{4y-5} + 3}{4\cdot \frac{3}{4y-5}} \cdot \frac{4y-5}{4y-5} = \frac{15 + 12y - 15}{12} = \frac{12y}{12} = y \).

Since \( y \) was chosen arbitrarily from the co-domain of \( f \), we’ve shown that \( f \) is onto.
(b) Let \( x \) and \( y \) be positive real numbers and suppose that \( f(x) = f(y) \). By the definition of \( f \), this translates into
\[
\frac{5x^2+3}{4x^2} = \frac{5y^2+3}{4y^2}
\]
So \( \frac{5}{4} + \frac{3}{4}x^2 = \frac{5}{4} + \frac{3}{4}y^2 \)
So \( \frac{3}{4}x^2 = \frac{3}{4}y^2 \)
So \( \frac{1}{x^2} = \frac{1}{y^2} \)
So \( x^2 = y^2 \).
Since \( x \) and \( y \) are known to be positive, this implies that \( x = y \), which is what we needed to show.

3. Pigeonhole Principle (12 points)

Use the pigeonhole principle to prove the following claim. See moodle for which version you have been assigned.

Version 1: Suppose “Breaking Bad: the Movie” will be shown in the open air tonight. There are 200 students from CS173 class who want to see the movie. So Prof. Rutenbar has reserved a 14 yard by 14 yard lawn for students to sit on. Students can sit anywhere within the reserved area.

Claim: there are at least two students in who will sit \( \leq \sqrt{2} \) yards apart.

Version 2: Prof. Windgrave is growing mold samples in his basement. He would like to grow 380 different samples, which need to be more than \( \sqrt{2} \) inches apart. He has a 19 inch by 19 inch sheet of agar. The samples do not need to be placed in any specific pattern: they just have to be far enough apart.

Claim: his sheet of agar isn’t large enough.

Hints for both versions: subdivide the square. What familiar geometrical feature might \( \sqrt{2} \) be measuring?

Solution Version 1: Consider mapping this square lawn to a typical x-y coordinate plane, with the square beginning at the origin (0,0) and extending to (14,14). Subdivide this square into 196 smaller square regions with area 1. Note that a square region means the interior of the region as well as its boundary (so regions are not all disjoint).

We need to choose 200 points in the larger square where the students will sit. By the Pigeonhole principle, with 200 choices to make over 14 \cdot 14 = 196 regions, at least two of these students must be part of the same square region.

Recall the Distance Formula for 2D points. If two points are part of the same square region of area 1, the maximum distance between them is \( \sqrt{2} \). Because we have said that there are at least two points that are part of the same square region, there must be two students with distance \( \sqrt{2} \)
Solution Version 2: Consider mapping the sheet of agar this square lawn to a typical x-y coordinate plane, with the square beginning at the origin (0,0) and extending to (19,19). Subdivide this square into 361 smaller square regions with area 1. Note that a square region means the interior of the region as well as its boundary (so regions are not all disjoint).

We need to choose 380 points in the larger square as locations for mold samples. By the Pigeonhole principle, with 380 choices to make over 19*19=361 regions, at least two of the samples must be in the same square region.

Recall the Distance Formula for 2D points. If two points are part of the same square region of area 1, the maximum distance between them is $\sqrt{2}$. Because we have said that there are at least two points that are part of the same square region, two of the mold samples must have been placed too close together. So the sheet of agar wasn’t large enough.