Problem 1: Checkbox (14 points)

Check the box that best characterizes each item. (2 points each)

\( \forall x \in \mathbb{Z}, \ \exists y \in \mathbb{N}, \ x \leq y \)
- true \( \checkmark \)
- false

\( f : \mathbb{N} \rightarrow \mathbb{R}, \ f(x) = x^2 + 2 \) is
- one-to-one but not onto \( \checkmark \)
- onto but not one-to-one
- neither one-to-one nor onto
- bijective
- not a valid function

\( f : \mathbb{N}^2 \rightarrow \mathbb{Z}, \ f(p, q) = 2^p3^q \) is
- one-to-one but not onto \( \checkmark \)
- onto but not one-to-one
- neither one-to-one nor onto
- bijective
- not a valid function

The chromatic number of a bipartite graph with at least one edge
- 1
- 2
- 3
- more than one possibility \( \checkmark \)

Number of connected components in \( C_{17} \)
- 0
- 1 \( \checkmark \)
- 2
- 17

Total number of leaves in a 3-ary tree of height \( h \):
- \( 3^h \)
- \( \frac{1}{2}(3^{h+1} - 1) \)
- \( \leq 3^h \) \( \checkmark \)
- \( 3^{h+1} - 1 \)

A path is a type of
- open walk \( \checkmark \)
- closed walk
Problem 2: Short answer (16 points)

(a) (8 points) If I have 8 potential players, how many different ways can I create two teams (unordered sets) of three players? The ordering of the teams themselves does not matter. Show/explain all work.

Solution: There are $8 \times 7 \times 6$ ways to choose the first three players in order, but then we need to divide by 3! ways to permute them, since the set of 3 is unordered. Likewise, there are $5 \times 4 \times 3$ ways to choose the next three players and 3! permutations of that set. Since the teams themselves are unordered, there are 2 ways to permute the teams. So there are \[
\frac{8 \times 7 \times 6 \times 5 \times 4 \times 3}{2 \times 3! \times 3!} = 8 \times 7 \times 5 \text{ different ways to create two teams of three players.}
\]

(b) (8 points) Consider the following grammar $G$

\[
S \rightarrow b a S \mid S S \\
S \rightarrow c \mid c a
\]

$S$ is the only valid start node. The valid terminal nodes are $a$, $b$, and $c$.

Here are two sequences of leaf labels. For each sequence, either draw a tree from grammar $G$ whose leaves have this sequence of labels, or else explain briefly why $G$ cannot generate this sequence of leaf labels.

\[
c a b a c
\]

Solution:

\[
S \\
/ \ \\
/ \ \\
/ \ \\
S \ S \\
/ \ /|\ \\
c \ a \ b \ a \ S \\
| \\
c
\]

\[
c a c b c
\]

Solution: This can’t be generated by $G$. The only rule in $G$ that produces a “b” also produces an “a” right after it. But this string has a “b” followed by only a “c”.

Problem 3: Graphs (16 points)

(a) (8 points) What is the chromatic number of graph G (below)? Justify your answer.

Solution: Upper bound: the coloring above shows that the chromatic number is no larger than 4. Lower bound: the chromatic number must be at least 4, because it contains a copy of $K_4$ (including the nodes B, C, D, and E). Since these upper and lower bounds are the same, the chromatic number be exactly 4.

(b) (8 points) Are the graphs shown below isomorphic? Justify your answer.

Solution: These graphs are not isomorphic. The righthand graph contains several 3-node cycles (e.g. G, C, D) but the lefthand graph doesn’t contain any.
Problem 4: Recursion Tree (18 points)

Use a recursion tree to find the closed form expression for the function $A$ defined by

$$T(1) = c$$

$$T(n) = 3T(n/3) + n$$

(a) Show a recursion tree with two levels of nodes below the root, labeled with the value for each node. It’s OK to label only one node per level. (4 points)

Solution: The first level contains one node labeled $n$. The second level contains three nodes labeled $n/3$. The third level contains nine nodes labeled $n/9$.

(b) For input value $n$, what is the level of the leaf nodes? (4 points)

Solution: $3^k = n$ so $k = \log_3 n$.

(c) For any non-leaf level $k$, what is the sum of values in the nodes? (3 points)

Solution: $3^k \cdot \frac{n}{3^k} = n$.

(d) What is the total value of the leaf nodes? (3 points)

Solution: $cn$

(e) What is the total value of all nodes, including all levels of the tree? (4 points)

Solution: $\sum_{i=0}^{\log_3 n-1} n + cn = n \log_3 n + cn$
Problem 5: Tree Induction (18 points)

A Pioneer tree is a binary tree whose nodes contain integers such that

- Every leaf node contains 5, 17, or 23.
- A node with one child contains the same number as its child.
- A node with two children contains the value \( x(y + 1) \), where \( x \) and \( y \) are the values in its children.

Use strong induction to prove that the value in the root of a Pioneer tree is always positive.

The induction variable is named \( h \) and it is the height of/in the tree.

Base Case(s): Solution: The smallest Pioneer trees consist of a single root node, which is also a leaf. By the definition of Pioneer tree, this must contain 5, 17, or 23, all of which are positive.

Inductive Hypothesis [Be specific, don’t just refer to “the claim”]:

Solution: Suppose that the root of a Pioneer tree of height \( h \) is always positive, for \( h = 0, \ldots, k - 1 \).

Inductive Step:

Solution: Let \( T \) be a Pioneer tree of height \( k \). There are two cases for what the top of \( T \) looks like.

Case 1: \( T \) consists of a root \( r \) with a single subtree \( S \) under it. \( r \) contains the same number as the root of \( S \). Since \( S \) must be shorter than \( k \), its root contains a positive number by the inductive hypothesis. Since \( r \) has the same label, \( r \) contains a positive number.

Case 2: \( T \) consists of a root \( r \) with a two subtrees \( S_1 \) and \( S_2 \). Suppose that the roots of \( S_1 \) and \( S_2 \) contain the numbers \( x \) and \( y \). Then, by the definition of Pioneer tree, \( r \) contains \( x(y + 1) \).

Since \( S_1 \) and \( S_2 \) are shorter than \( k \), \( x \) and \( y \) must be positive by the inductive hypothesis. Since \( y \) is positive, so is \( y + 1 \). Since \( x \) and \( y + 1 \) are positive, so is \( x(y + 1) \). So the root of \( T \) contains a positive number.

So, in either case, the root of \( T \) contains a positive number.
Problem 6: Induction (18 points)

Using strong induction, prove that the following holds for all positive integers \( p \) and \( b \):

(Correction during the exam: assume \( b > 1 \).)

\[
\sum_{j=0}^{p} b^j = \frac{1 - b^{p+1}}{1 - b}
\]

Solution:

The induction variable is \( p \).

Base case(s): \( p = 1 \). Then \( \sum_{j=0}^{p} b^j + b \) and \( \frac{1 - b^{p+1}}{1 - b} = \frac{1 - b^2}{1 - b} = 1 + b \)

Inductive hypothesis [Be specific, don’t just refer to “the claim”]: Suppose that \( \sum_{j=0}^{p} b^j = \frac{1 - b^{p+1}}{1 - b} \) for any \( b > 1 \) and for \( p = 1, 2, \ldots, k \).

Rest of the inductive step: By the inductive step, we know that \( \sum_{j=0}^{k} b^j = \frac{1 - b^{k+1}}{1 - b} \)

So then

\[
\sum_{j=0}^{k+1} b^j = b^{k+1} + \sum_{j=0}^{k} b^j = b^{k+1} + \frac{1 - b^{k+1}}{1 - b}
\]

\[
= \frac{1}{1 - b} \cdot [(1 - b)b^{k+1} + (1 - b^{k+1})]
\]

\[
= \frac{1}{1 - b} \cdot [b^{k+1} - b^{k+2} + 1 - b^{k+1}]
\]

\[
= \frac{1}{1 - b} \cdot [1 - b^{k+2}] = \frac{1 - b^{k+2}}{1 - b}
\]