1. **Nested Quantifiers**

Prove or disprove the statements in (a), (b), and (d). **Hint:** these proofs/disproofs are meant to be very brief.

(a) \( \exists x \in \mathbb{N}, \forall y \in \mathbb{N}, \gcd(x, y) = 1 \)

(b) \( \forall x \in \mathbb{Z}^+, \exists y \in \mathbb{Z}^+, x = y^2 \)

(c) Suppose that \( f \) is a function from \( \mathbb{Z}_6 \) to \( \mathbb{Z}_8 \), and \( \exists c \in \mathbb{Z}_8, \forall x \in \mathbb{Z}_6, f(x) = c. \)

Give a one sentence description of the function \( f \).

(d) \( \exists f : \mathbb{Z}_6 \rightarrow \mathbb{Z}_8, \exists c \in \mathbb{Z}_8, \forall x \in \mathbb{Z}_6, f(x) = c \)

**Function Proofs**

(a) Suppose that \( g : A \rightarrow B \) and \( f : B \rightarrow C \). Prof. Snape claims that if \( f \circ g \) is onto, then \( g \) is onto. Disprove this claim using a concrete counter-example in which \( A, B, \) and \( C \) are all small finite sets.

(b) Suppose that \( g : \mathbb{Z} \rightarrow \mathbb{Z} \) is one-to-one. Let’s define the function \( f : \mathbb{Z} \rightarrow \mathbb{Z}^2 \) by \( f(x) = (x^2, g(x)) \). Prove that \( f \) is one-to-one.

(c) Define the function \( f \) as follows:

- \( f(1) = 1 \)
- \( f(2) = 5 \)
- \( f(n + 1) = 5f(n) - 6f(n - 1) \)

Suppose we’re proving that \( f(n) = 3^n - 2^n \) for every positive integer \( n \). State the inductive hypothesis and the conclusion of the inductive step.
Induction

Let the function \( f : \mathbb{N} \to \mathbb{Z} \) be defined by
\[
\begin{align*}
    f(0) &= 1 \\
    f(1) &= 6 \\
    \forall n \geq 2, \quad f(n) &= 6f(n - 1) - 9f(n - 2)
\end{align*}
\]

Use strong induction on \( n \) to prove that \( \forall n \geq 0, \ f(n) = (1 + n)3^n \).

Base case(s):

Inductive hypothesis [Be specific, don’t just refer to “the claim”]:

Rest of the inductive step:
(a) How many connected components does each graph have?
(b) Are graphs G1 and G2 (above) isomorphic? Briefly justify your answer.
(c) What is the diameter of G3?
(d) Does G3 contain an Euler circuit? Why or why not?
(e) Does G2 and/or G3 contain a cut edge? If so, identify which edge(s) are cut edges.
(f) How many isomorphisms are there from G3 to G3? Justify your answer or show work.