1. Nested Quantifiers

Prove or disprove the statements in (a), (b), and (d). **Hint:** these proofs/disproofs are meant to be very brief.

(a) \( \exists x \in \mathbb{N}, \forall y \in \mathbb{N}, \text{GCD}(x, y) = 1 \)

**Solution:** True. Let \( x = 1 \). Note that GCD(1, y) = 1 for any choice of y since 1 divides all natural numbers (including 0).

(b) \( \forall x \in \mathbb{Z}^+, \exists y \in \mathbb{Z}^+, x = y^2 \)

**Solution:** This says that all positive integers are perfect squares, which is false. Choose \( x = 2 \). If there were an integer \( y \) such that \( 2 = y^2 \), then, \( y = \sqrt{3} \) must be an integer, which is absurd.

(c) Suppose that \( f \) is a function from \( \mathbb{Z}_6 \) to \( \mathbb{Z}_8 \), and \( \exists c \in \mathbb{Z}_8, \forall x \in \mathbb{Z}_6, f(x) = c \).

**Solution:** The function \( f \) sends all inputs to a single output \( c \in \mathbb{Z}_8 \), i.e., it is a constant function.

(d) \( \exists f : \mathbb{Z}_6 \to \mathbb{Z}_8, \exists c \in \mathbb{Z}_8, \forall x \in \mathbb{Z}_6, f(x) = c \)

**Solution:** True. Let \( c = [0] \) and simply take \( f \) to be the constant function which sends all inputs \( x \in \mathbb{Z}_6 \) to \([0] \in \mathbb{Z}_8 \), i.e., \( f(x) = [0] \) for all \( x \in \mathbb{Z}_6 \).

**Function Proofs**

(a) Suppose that \( g : A \to B \) and \( f : B \to C \). Prof. Snape claims that if \( f \circ g \) is onto, then \( g \) is onto. Disprove this claim using a concrete counter-example in which \( A \), \( B \), and \( C \) are all small finite sets.

**Solution:** Suppose that \( A = \{1, 2\} \), \( B = \{3, 4, 5\} \), and \( C = \{\text{red, blue}\} \). Define \( g \) by \( g(1) = 3 \) and \( g(2) = 5 \). Define \( f \) by \( f(3) = \text{red}, f(4) = \text{red}, \) and \( f(5) = \text{blue} \). Then \( (f \circ g)(1) = \text{red} \) and \( (f \circ g)(2) = \text{blue} \). So \( f \circ g \) is onto because every element of \( C \) has a pre-image. However, \( g \) isn’t onto because no element of \( A \) maps onto 4.

(b) Suppose that \( g : \mathbb{Z} \to \mathbb{Z} \) is one-to-one. Let’s define the function \( f : \mathbb{Z} \to \mathbb{Z}^2 \) by \( f(x) = (x^2, g(x)) \). Prove that \( f \) is one-to-one.

**Solution:** Let \( x \) and \( y \) be integers. Suppose that \( f(x) = f(y) \). By the definition of \( f \), this means that \( (x^2, g(x)) = (y^2, g(y)) \). So then \( x^2 = y^2 \) and \( g(x) = g(y) \). Since \( g(x) = g(y) \) and \( g \) is one-to-one, \( x = y \). So we have that \( f(x) = f(y) \) implies \( x = y \). This means that \( f \) is one-to-one.

(c) Define the function \( f \) as follows:

- \( f(1) = 1 \)
- \( f(2) = 5 \)
• \( f(n + 1) = 5f(n) - 6f(n - 1) \)

Suppose we’re proving that \( f(n) = 3^n - 2^n \) for every positive integer \( n \). State the inductive hypothesis and the conclusion of the inductive step.

**Solution:** Inductive hypothesis: suppose that \( f(n) = 3^n - 2^n \) for \( n = 1, 2, \ldots k \), for some integer \( k \).

Conclusion of the inductive step: \( f(k + 1) = 3^{k+1} - 2^{k+1} \).

Note 1: a strong hypothesis is required because the formula reaches back two integers.

Note 2: the variable \( k \) in the conclusion matches the upper bound in the hypothesis. A common mistake is to have it match the variable in the hypothesis equation \( (n) \). We’re assuming that the equation holds for all values up through \( k \), so we need to prove it holds for \( k + 1 \).

**Induction**

Let the function \( f : \mathbb{N} \to \mathbb{Z} \) be defined by

\[
\begin{align*}
f(0) &= 1 \\
f(1) &= 6 \\
\forall n \geq 2, f(n) &= 6f(n - 1) - 9f(n - 2)
\end{align*}
\]

Use strong induction on \( n \) to prove that \( \forall n \geq 0, f(n) = (1 + n)3^n \).

Base case(s):

**Solution:** \( f(0) = 1 = (1 + 0)3^0 \) and \( f(1) = 6 = (1 + 1)3^1 \). We need to check two base cases because the inductive step will reach back two integers.

Inductive hypothesis [Be specific, don’t just refer to “the claim”]:

**Solution:** Suppose that \( f(n) = (1 + n)3^n \) for \( n = 0, 1, \ldots, k \), for some \( k \geq 1 \).

Rest of the inductive step:

**Solution:** \( f(k + 1) = 6f(k) - 9f(k - 1) \) by the definition of \( f \). By the inductive hypothesis, we know that \( f(k) = (1 + k)3^k \) and \( f(k - 1) = k3^{k-1} \). So by substituting, we get

\[
\begin{align*}
f(k + 1) &= 6(1 + k)3^k - 9k3^{k-1} \\
&= 2(1 + k)3^{k+1} - k3^{k+1} \\
&= 2 \cdot 3^{k+1} + k3^{k+1} - k3^{k+1} \\
&= 2 \cdot 3^{k+1} + k3^{k+1} \\
&= (k + 2)3^{k+1}
\end{align*}
\]

So \( f(k + 1) = (k + 2)3^{k+1} \), which is what we needed to show.
(a) How many connected components does each graph have?
Solution: G1 has two connected components. G2 and G3 each have one connected component.

(b) Are graphs G1 and G2 (above) isomorphic? Briefly justify your answer.
Solution: No. G2 is connected, but G1 isn’t connected. Also, G2 contains a cycle with 6 vertices, and G1 doesn’t.

(c) What is the diameter of G3?
Solution: 4. (It’s the number of edges on a shortest path between the two vertices furthest apart. In this case, y and either q or r.)

(d) Does G3 contain an Euler circuit? Why or why not?
Solution: No, it can’t contain an Euler circuit because some of the vertices (e.g. p) have odd degree.

(e) Does G2 and/or G3 contain a cut edge? If so, identify which edge(s) are cut edges.
Solution: G3 contains a cut edge: the edge connecting p and s. G2 does not contain a cut edge.

(f) How many isomorphisms are there from G3 to G3? Justify your answer or show work.
Solution: p is the only degree-3 node which is connected to two degree-2 nodes. So p must map to itself. Similarly, s must map to itself because it’s the only node whose neighbors all have degree 3.
However, r and q can be interchanged without changing the graph structure.
We can also interchange t and w without changing the graph structure.
So we have $2 \times 2 = 4$ isomorphisms of G3 to itself.