The Last Lecture

A Brief Encounter with the Uncomputable

Lecture 26

Question

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- Which of the following are countably infinite?
  1. Set of all prime numbers
  2. Set of all bit strings of length 32
  3. Set of all bit strings of finite length
  4. Set of all infinitely long bit strings

  A. 1, 2, 3 and 4
  B. 1, 2 and 3
  C. 1 and 4
  D. 1 and 3
  E. None of the above choices

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Contradiction!

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There is an onto function from A to B iff there is a one-to-one function from B to A

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  - This statement must be true if theory <u>consistent</u> (else a false statement is provable). Then the theory would be <u>incomplete</u>.

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  - Because $|\mathbb{T}^2|=|\mathbb{T}|$ (bijection by interleaving), and we saw $|\mathbb{R}|=|\mathbb{T}|$ (and hence $|\mathbb{R}^2|=|\mathbb{T}^2|$ too)

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    shown uncomputabe by Church and Turing [1936]

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  - Hilbert's 10th problem: find an algorithm to check if a "Diophantine equation" has a solution (i.e., check if there is an integer solution to all the variables in a polynomial. e.g., the ones in Fermat's last theorem, $x^3+y^3=z^3$, $x^4+y^4=z^4$, ...)
  - Hilbert's Entscheidungsproblem: given a statement in first order logic, check if it is true/provable
  - The Halting Problem: Given a (program,input) pair decide if the program halts or not

shown uncomputable in 1970

shown uncomputabe by Church and Turing [1936]

The Uncomputable

- "Almost all" functions are uncomputable!
  - Later (CS 373): distinction between computing a predicate ("deciding") and "recognizing" when the predicate is true
  - Almost all predicates "unrecognizable" too
- But do we care about all these functions? Often yes!
  - Hilbert's 10th problem: find an algorithm to check if a "Diophantine equation" has a solution (i.e., check if there is an integer solution to all the variables in a polynomial. e.g., the ones in Fermat's last theorem, $x^3+y^3=z^3$, $x^4+y^4=z^4$, ...)

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    Turing [1936]

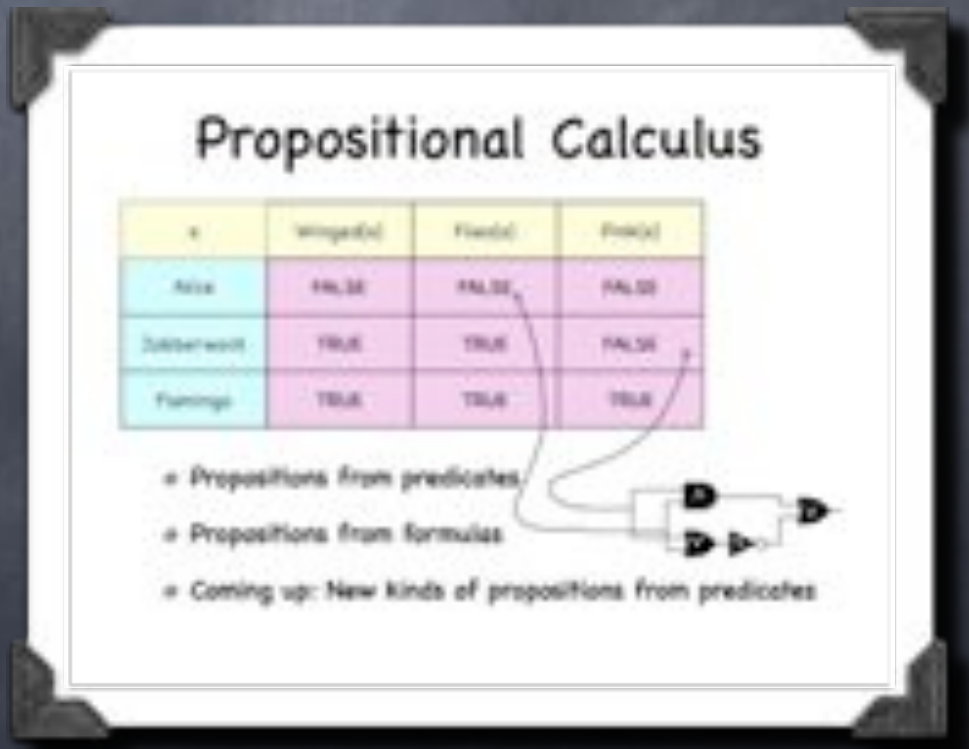
Discrete Structures

Wrap Up!

Discrete Structures (Intro to Mind Bending)

Wrap Up!

We Learned a Lot! :-)



<u>Basic tools for expressing ideas</u>

Logic, Proofs,

Sets, Relations, Functions

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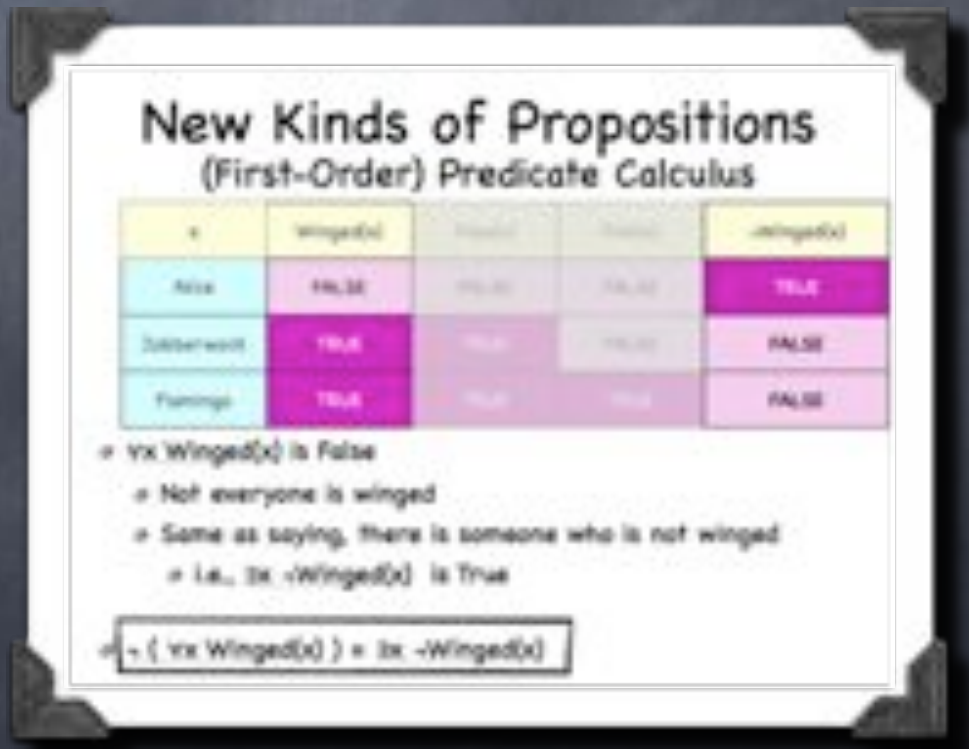

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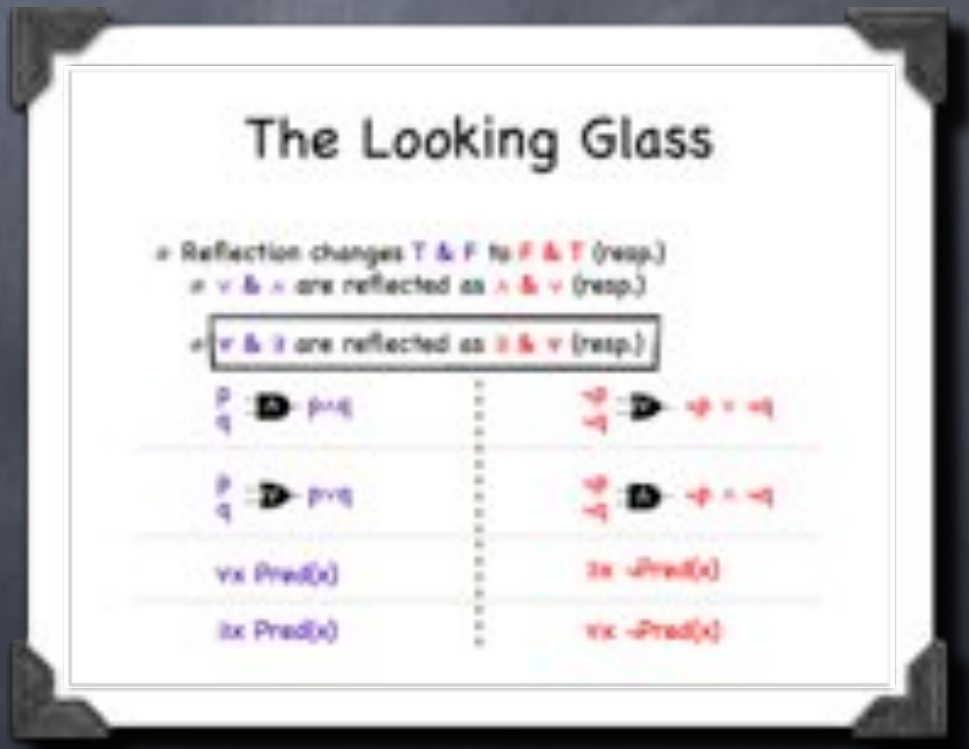

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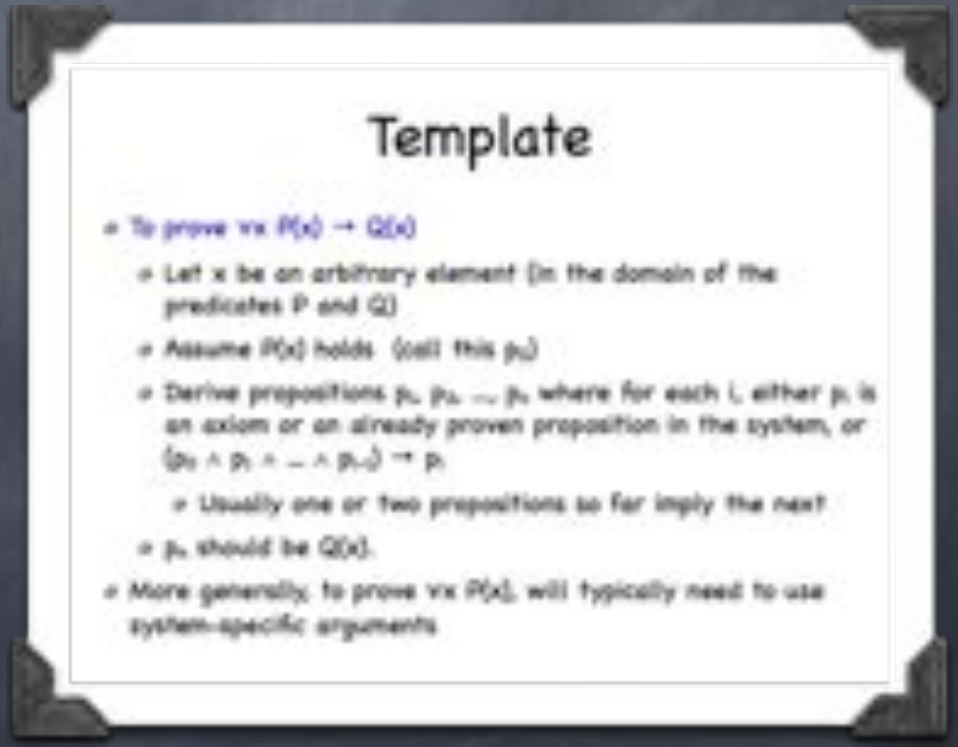

Template

- To prove $\forall x\ P(x) \rightarrow Q(x)$
  - Let x be an arbitrary element (in the domain of the predicates P and Q)
  - Assume P(x) holds (call this p_0)
  - Derive propositions $p_0, p_1, \ldots, p_n$ where for each i, either p_i is an axiom or an already proven proposition in the system, or $(p_0 \wedge p_1 \wedge \ldots \wedge p_{i-1}) \rightarrow p_i$
    - Usually one or two propositions so far imply the next
  - p_n should be Q(x).
- More generally, to prove $\forall x\ P(x)$, will typically need to use system-specific arguments

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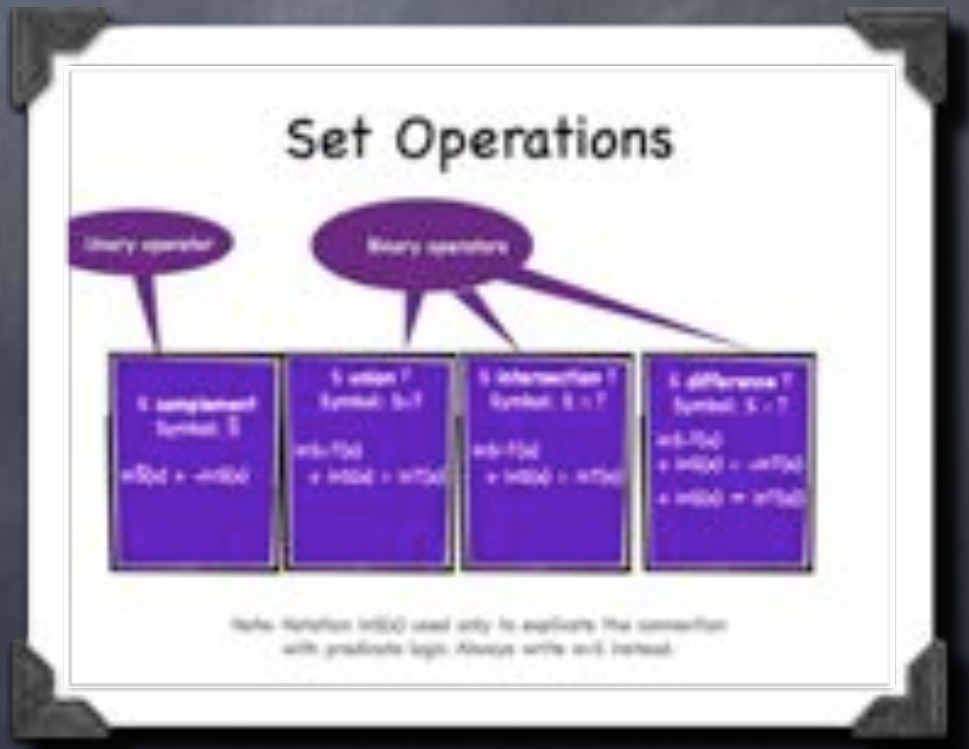

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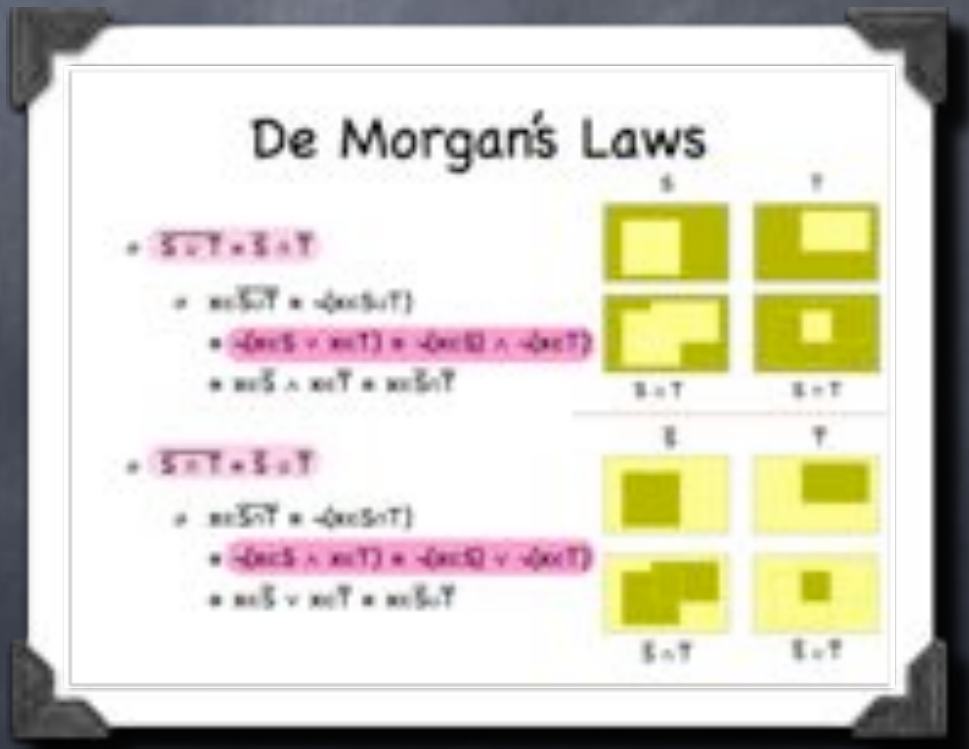

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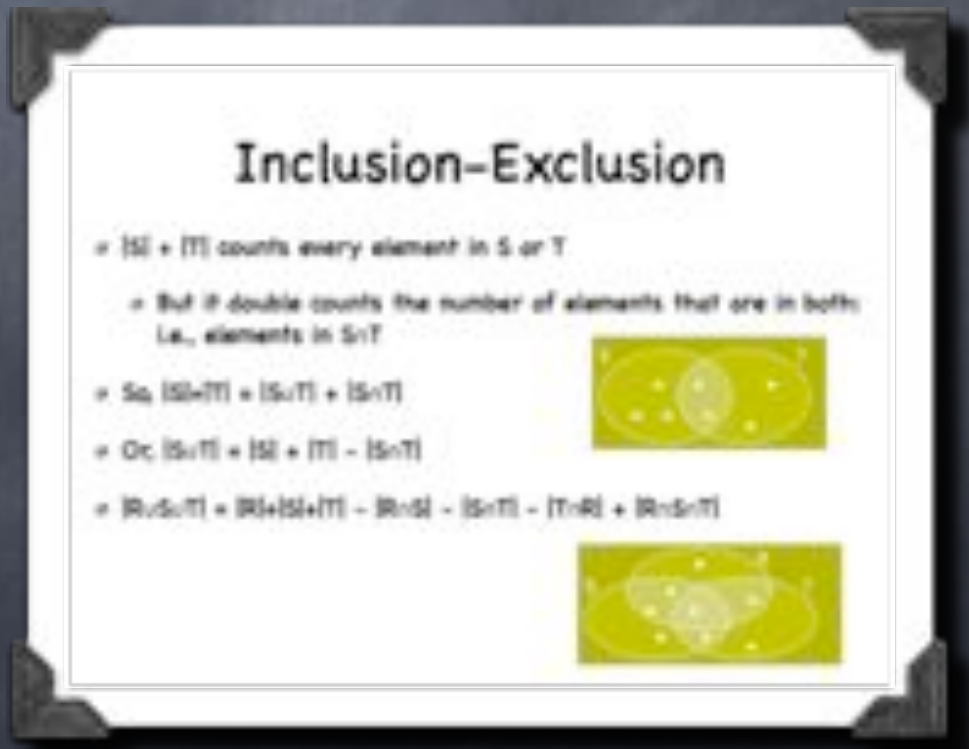

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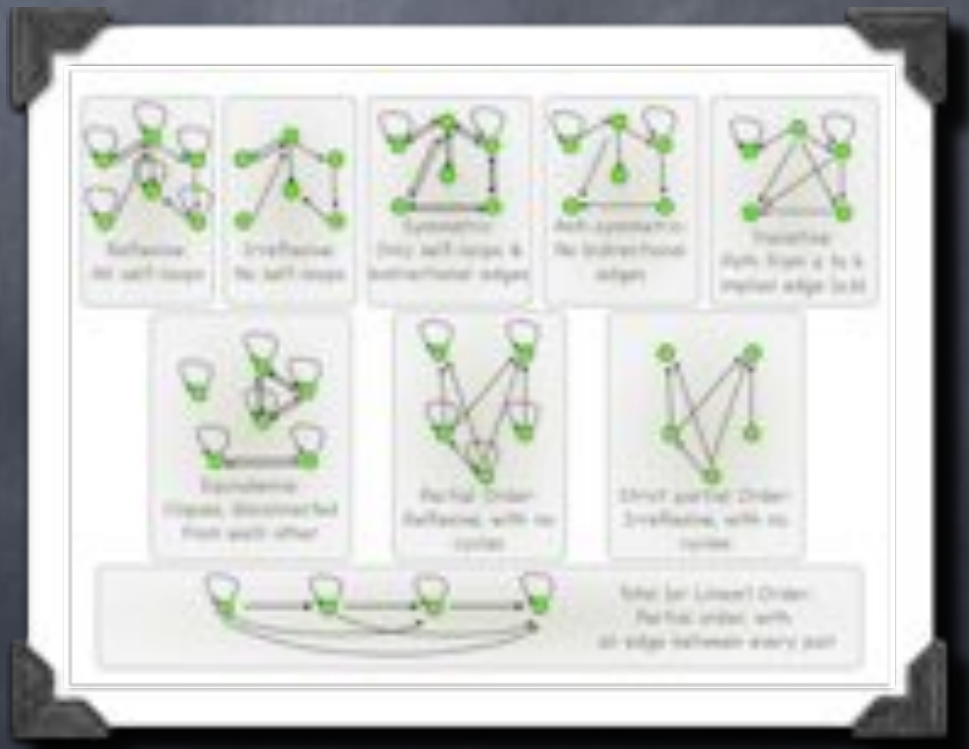

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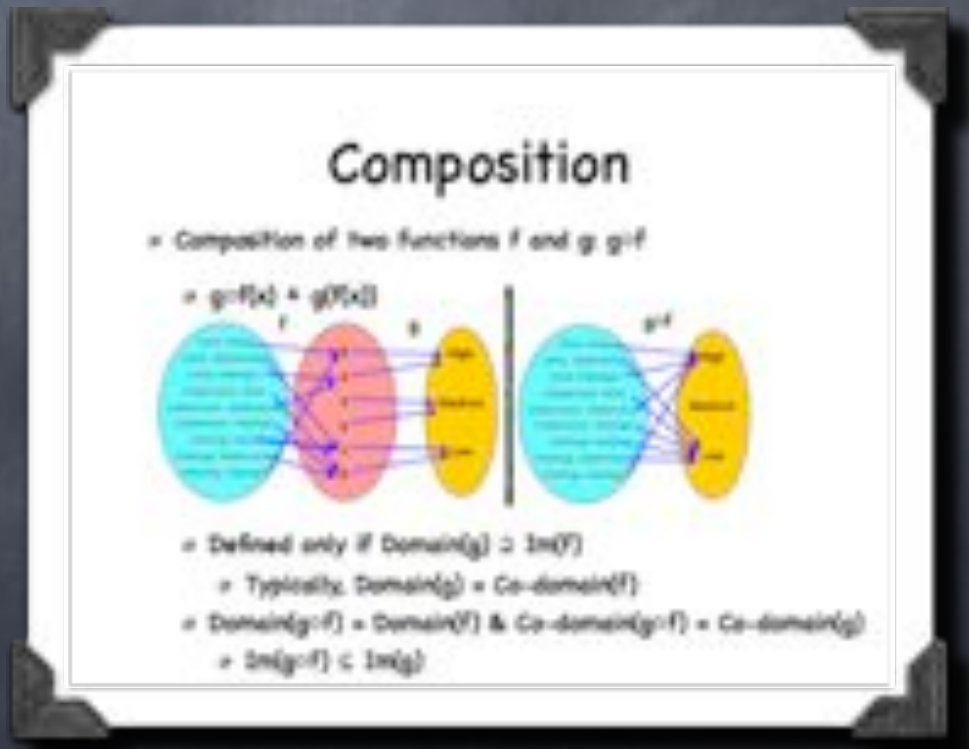

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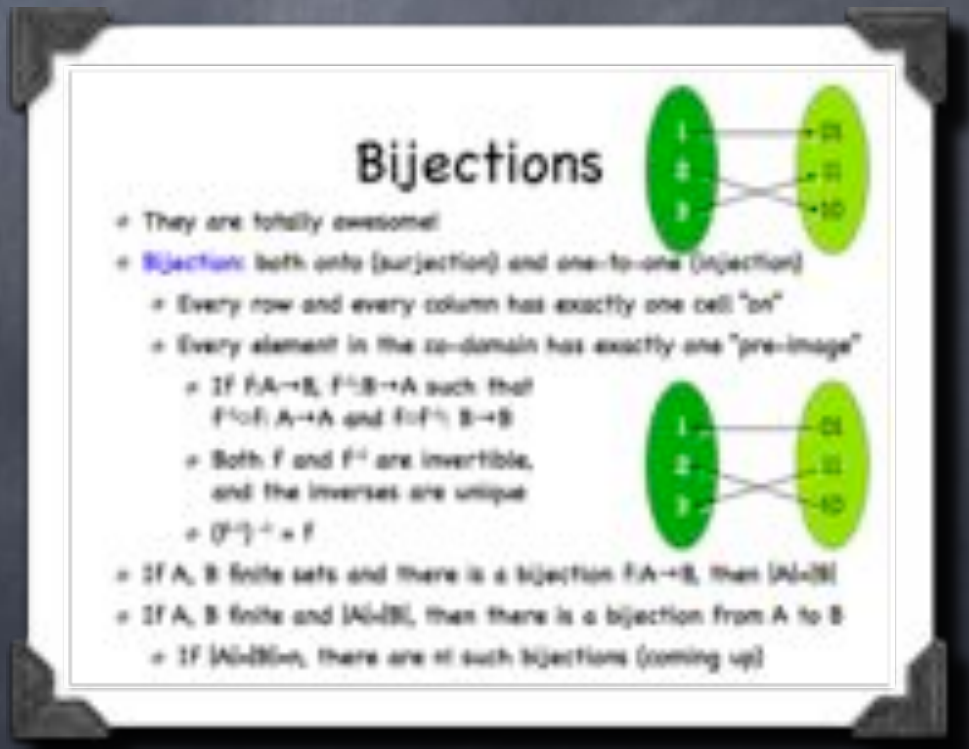

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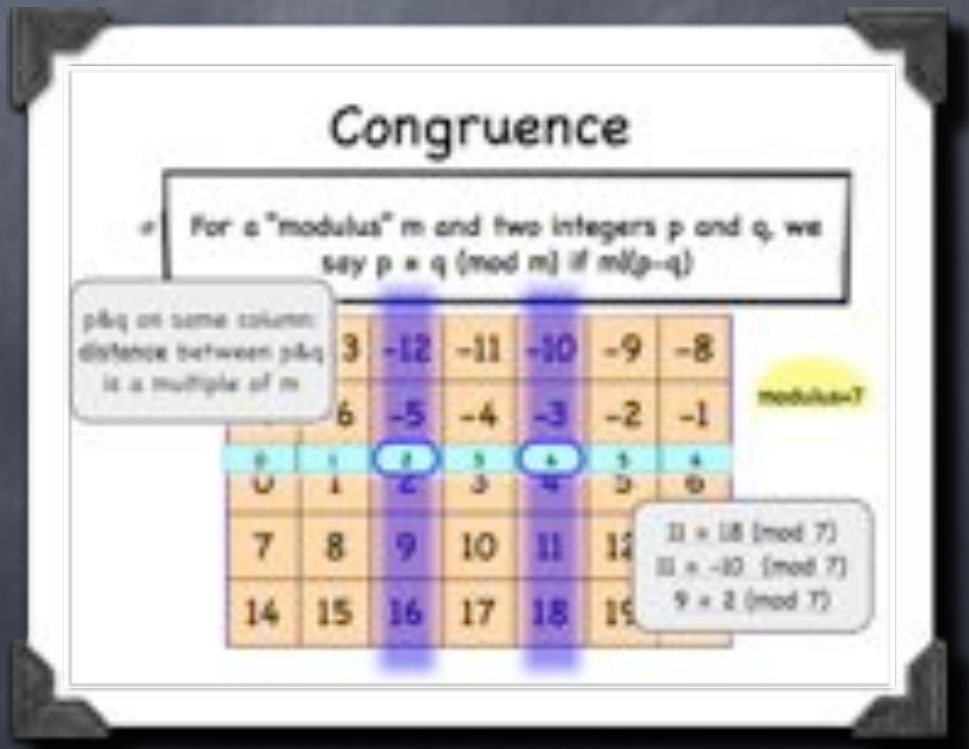

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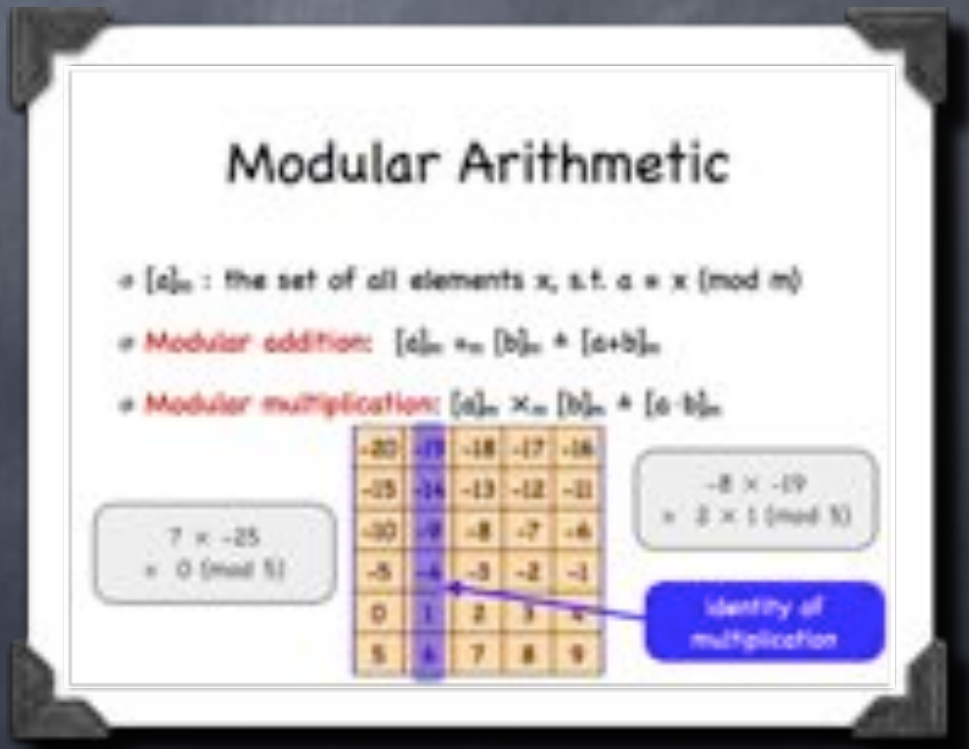

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Numbers

Graphs

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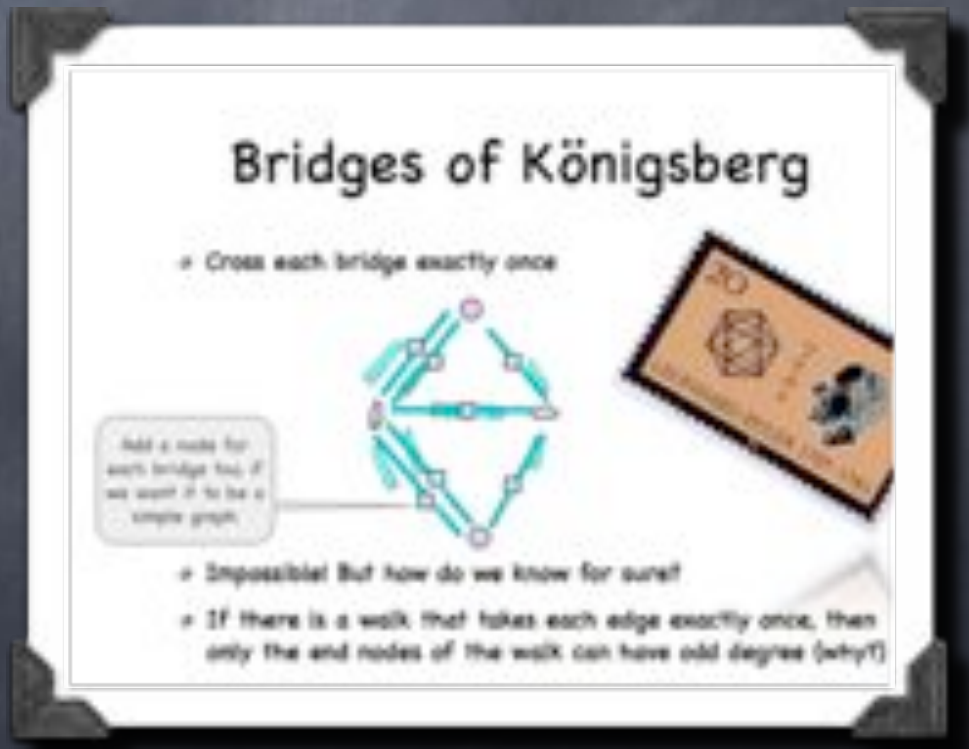

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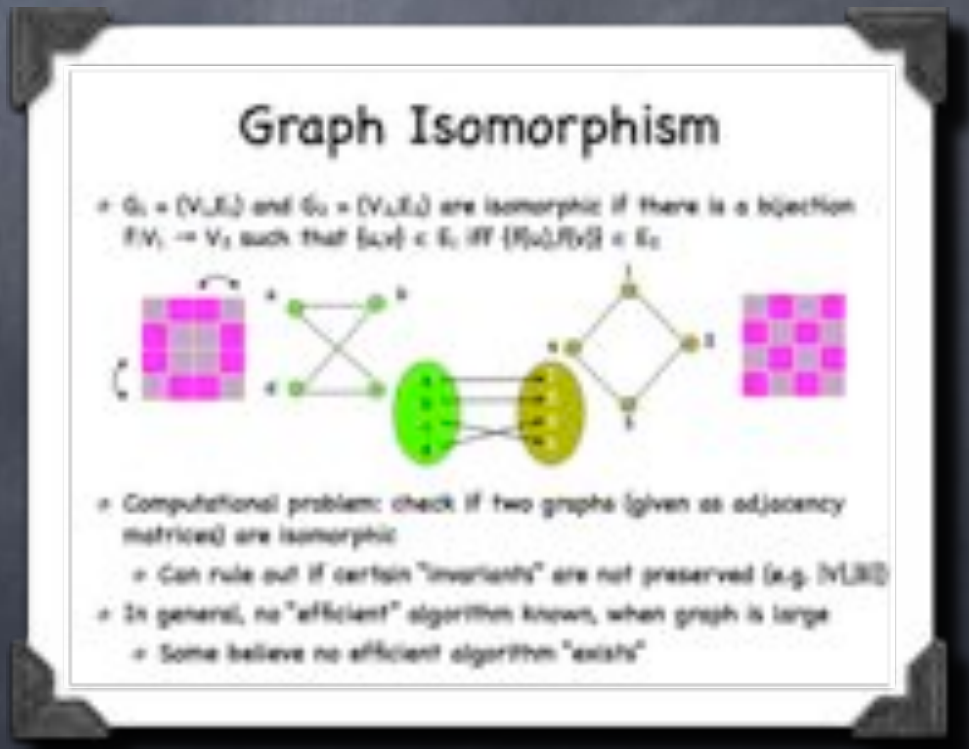

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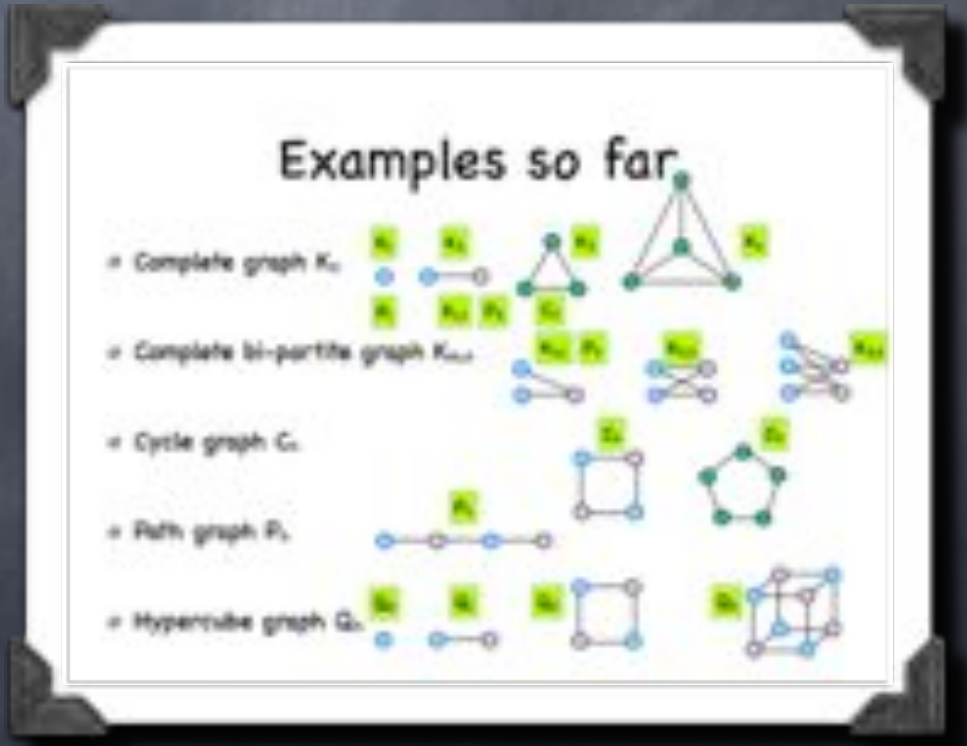

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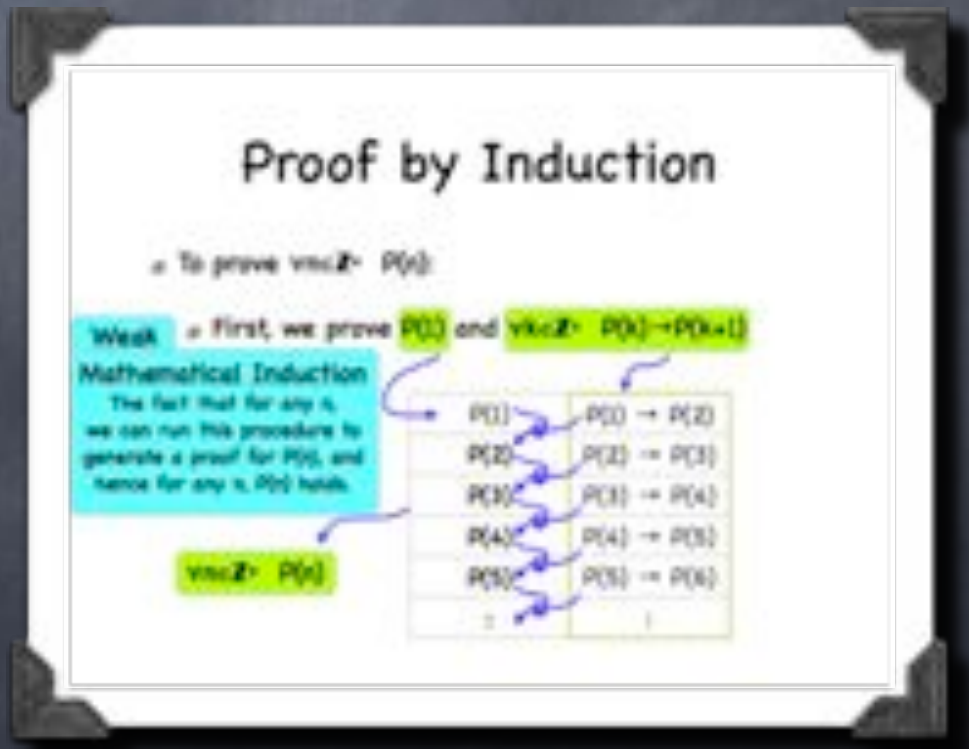

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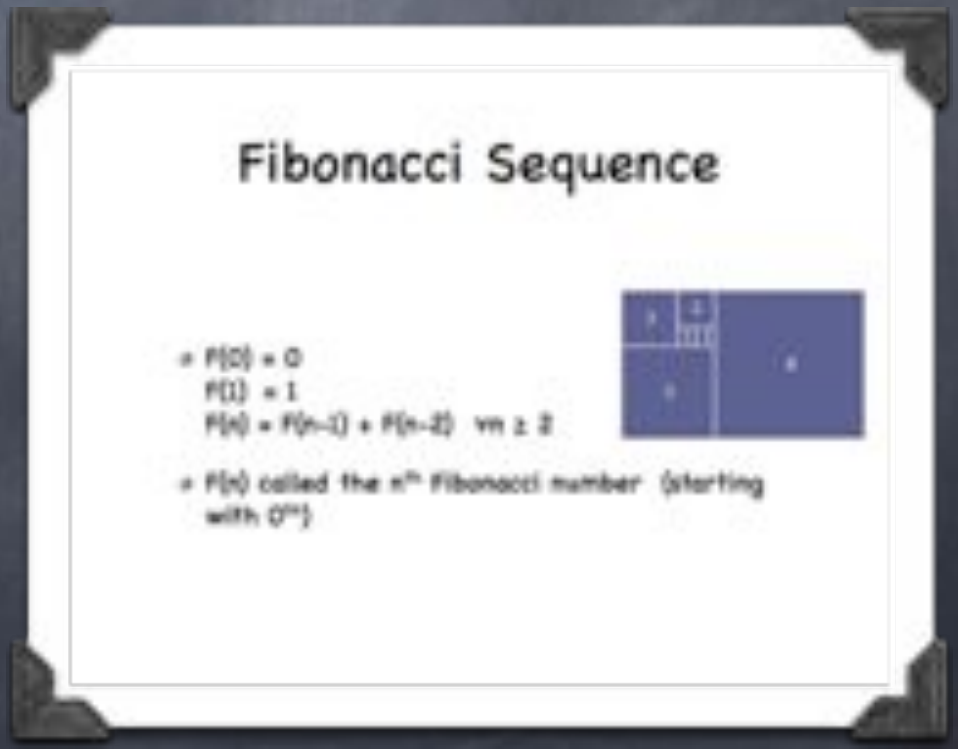

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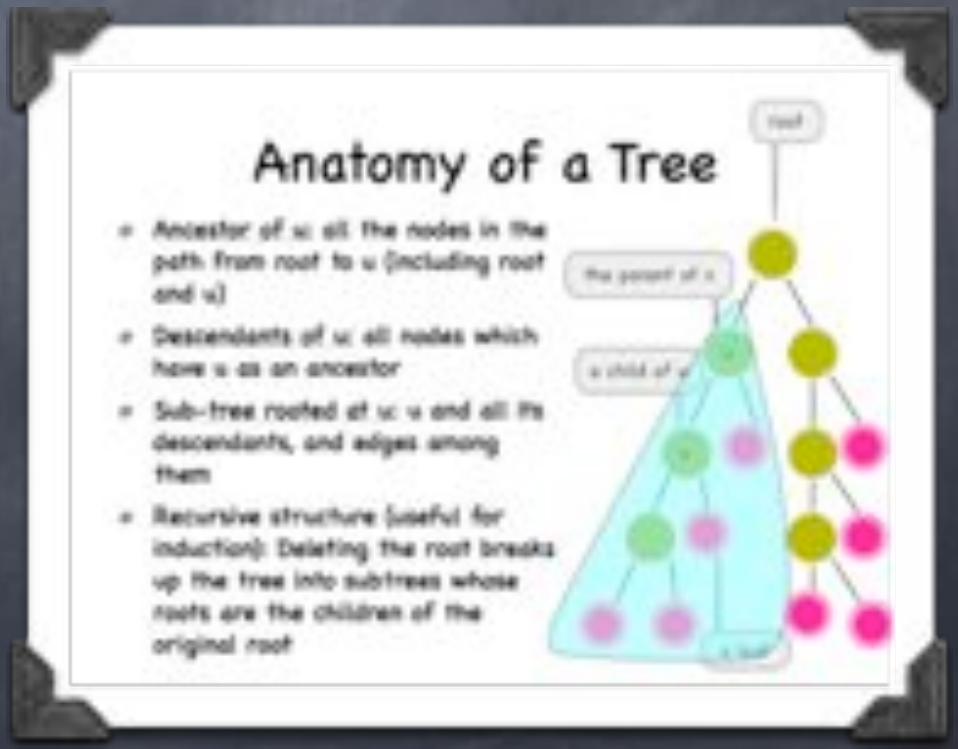

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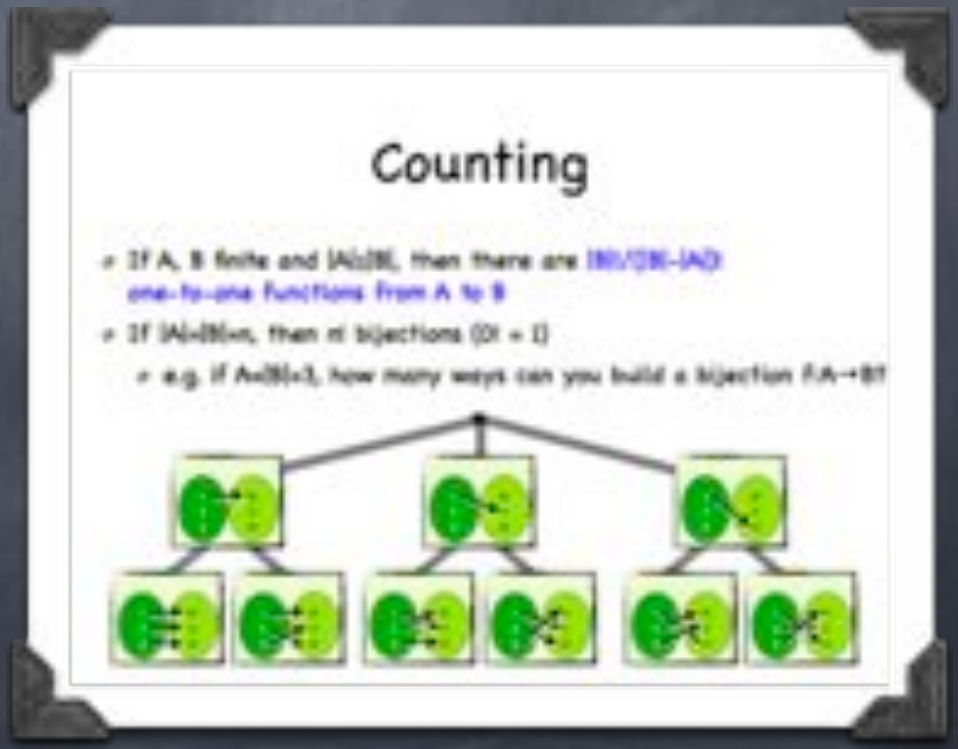

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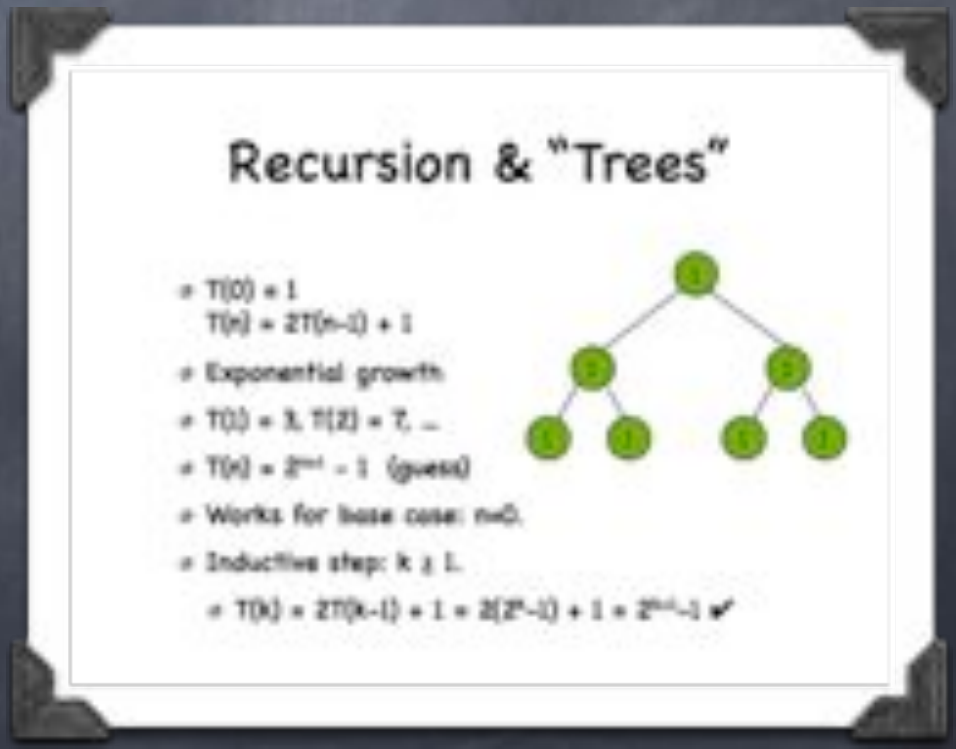

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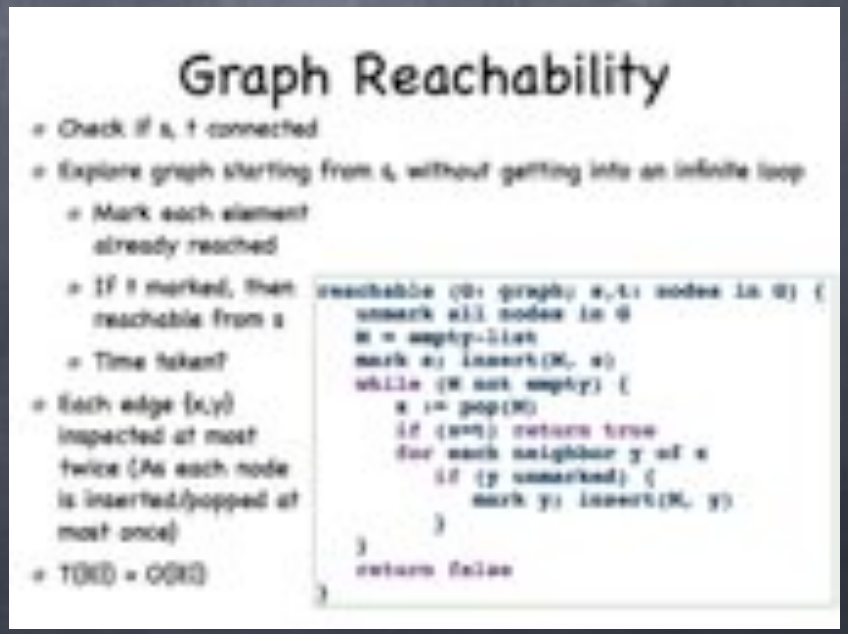

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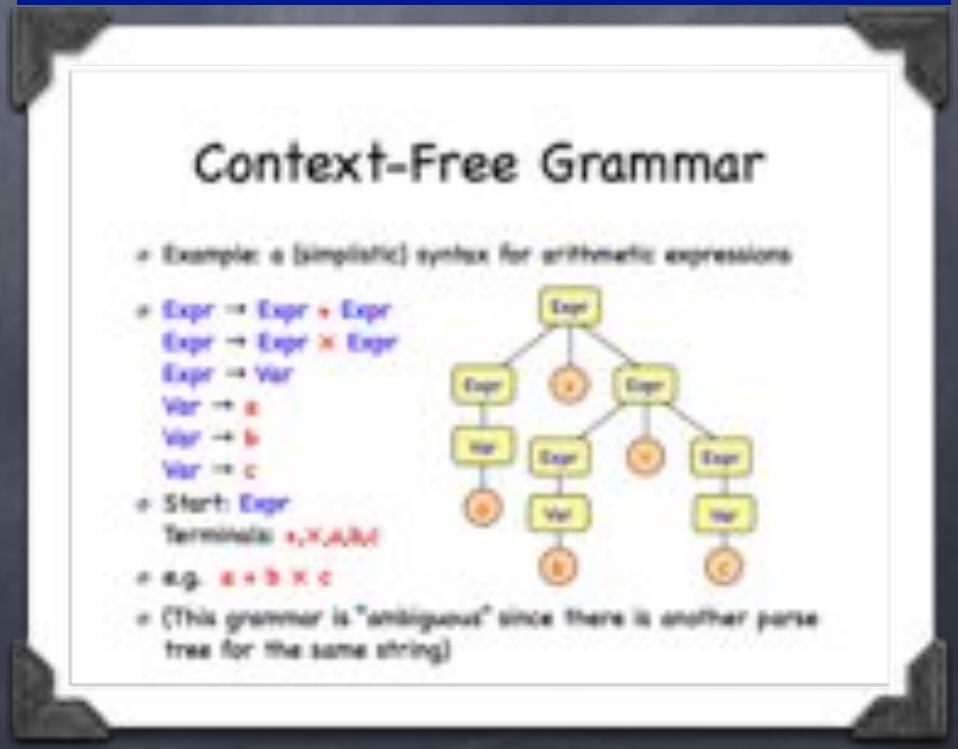

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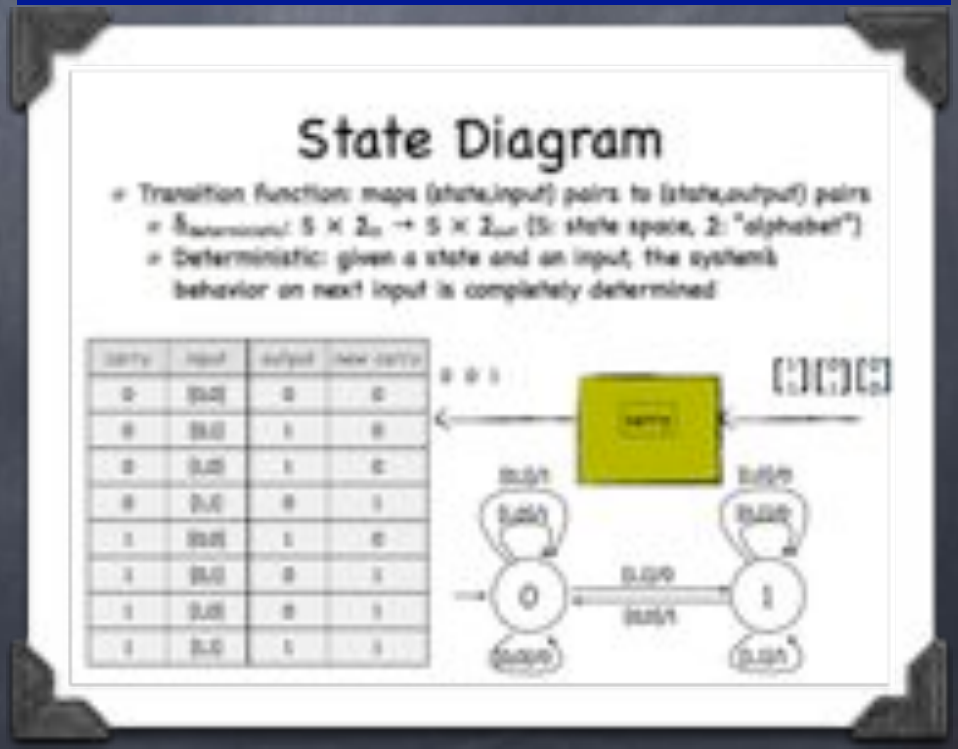

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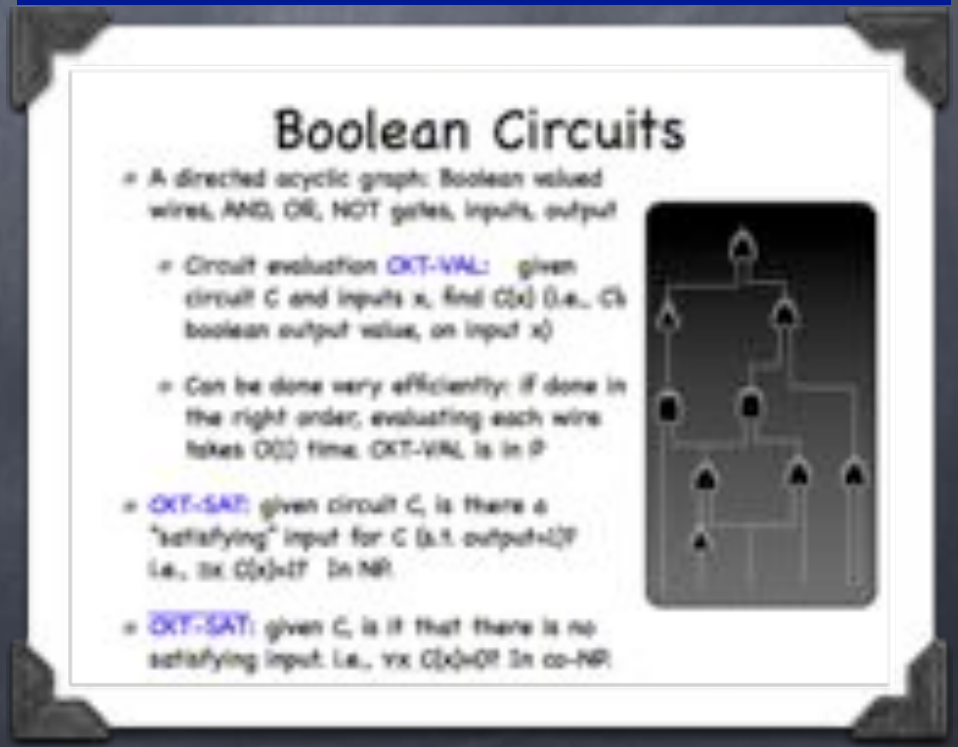

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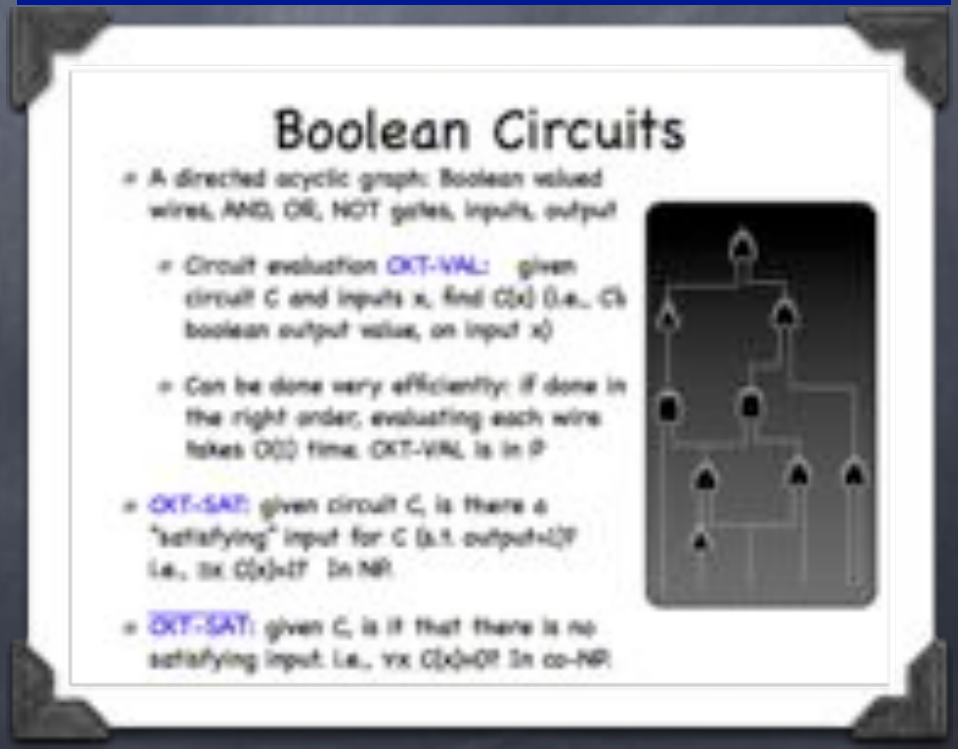

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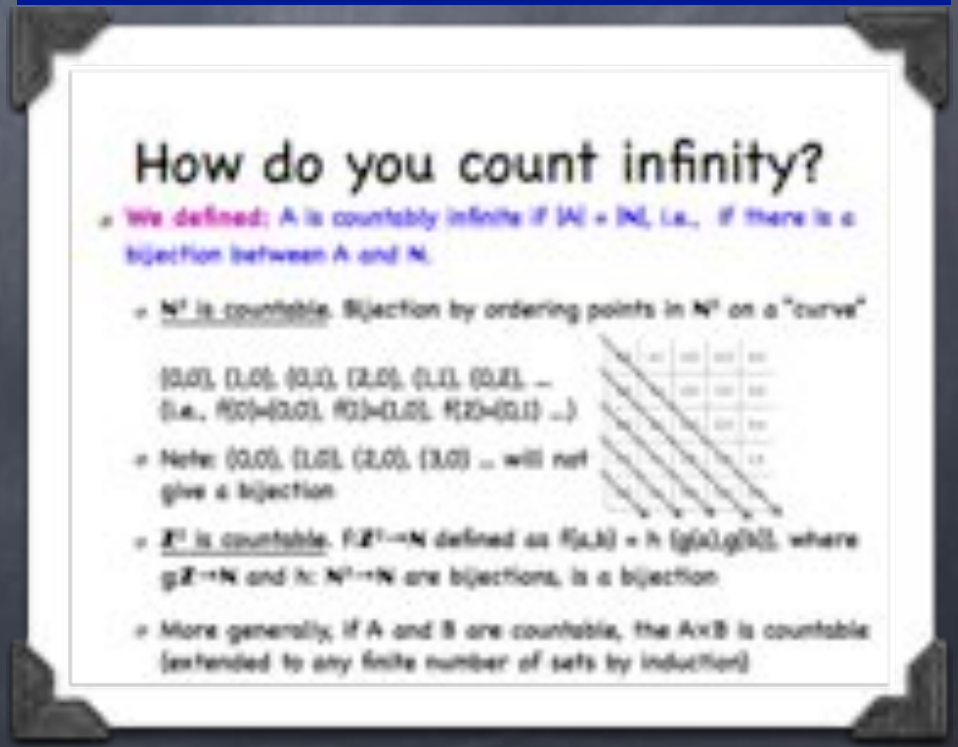

How do you count infinity?

- We defined: A is countably infinite if $|A| = |\mathbb{N}|$, i.e., if there is a bijection between A and $\mathbb{N}$.
  - $\mathbb{N}^2$ is countable. Bijection by ordering points in $\mathbb{N}^2$ on a "curve"

    (0,0), (1,0), (0,1), (2,0), (1,1), (0,2), ...
    (i.e., f(0)=(0,0), f(1)=(1,0), f(2)=(0,1) ...)
  - Note: (0,0), (1,0), (2,0), (3,0) ... will not give a bijection
  - $\mathbb{Z}^2$ is countable. $f: \mathbb{Z}^2 \to \mathbb{N}$ defined as $f(a,b) = h(g(a),g(b))$, where $g: \mathbb{Z} \to \mathbb{N}$ and $h: \mathbb{N}^2 \to \mathbb{N}$ are bijections, is a bijection
  - More generally, if A and B are countable, the $A \times B$ is countable (extended to any finite number of sets by induction)

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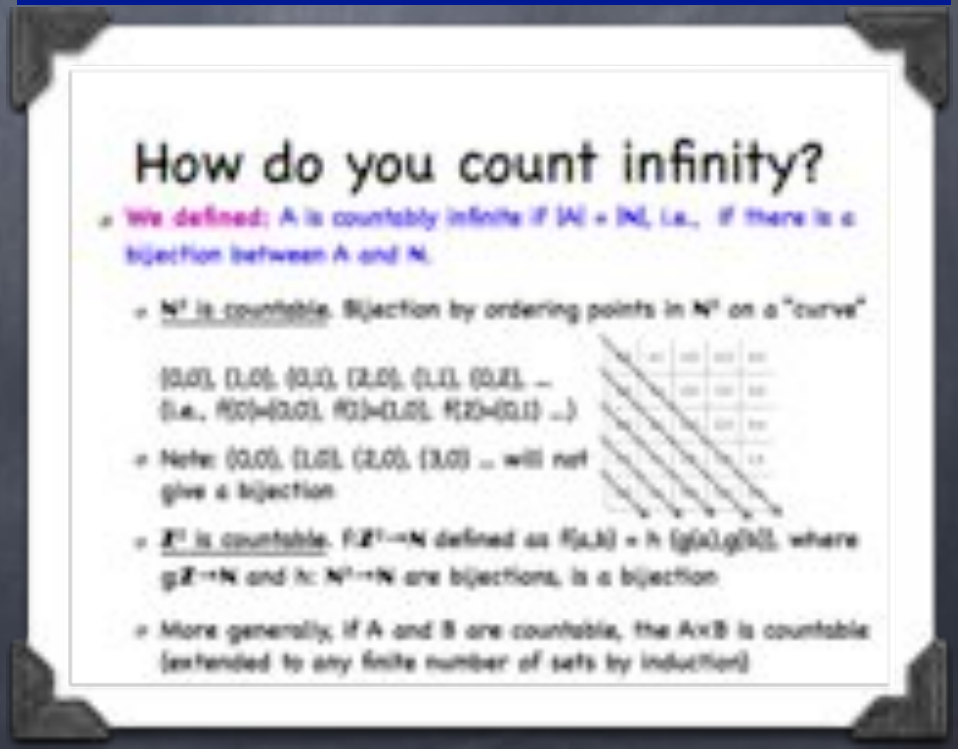

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- Now: ICES forms