# Countability

Lecture 25

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  - $|\mathbb{N}| = |2\mathbb{Z}|$. $h: \mathbb{N}\to 2\mathbb{Z}$ defined as $h = f\circ g^{-1}$

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- Intuition: all "discrete" sets are countable

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| | | | | |
|---|---|---|---|---|
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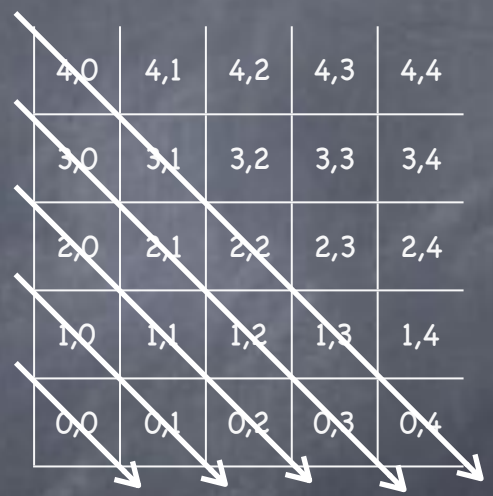

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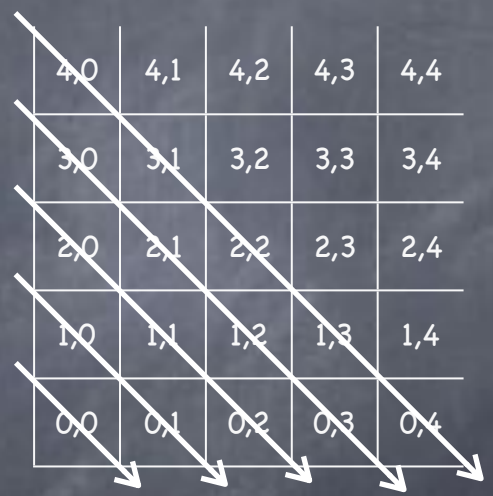


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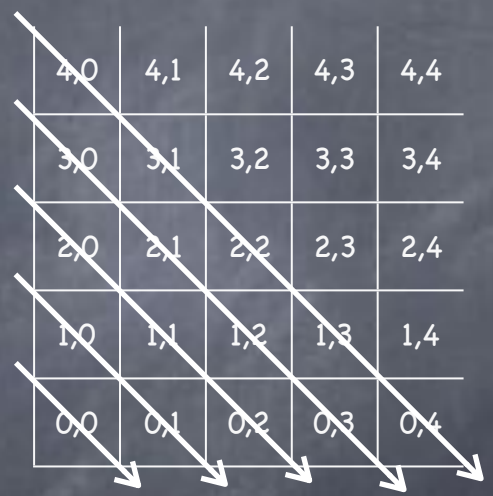


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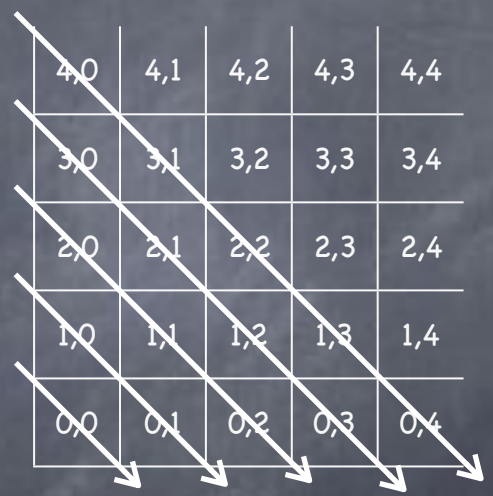


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  - More generally, if A and B are countable, the $A \times B$ is countable (extended to any finite number of sets by induction)

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  - Also can construct a one-to-one function $h:\mathbb{N}\rightarrow\mathbb{Q}$ as $h(a)=a$

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- Want to show $|\mathbb{Q}|=|\mathbb{N}|$ (i.e., a bijection between $\mathbb{Q}$ and $\mathbb{N}$)

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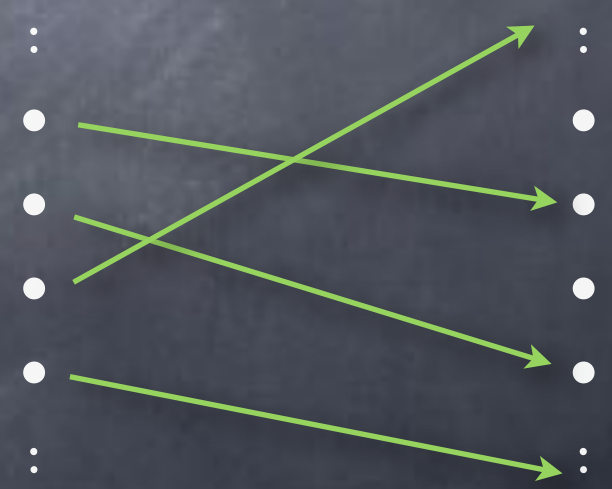

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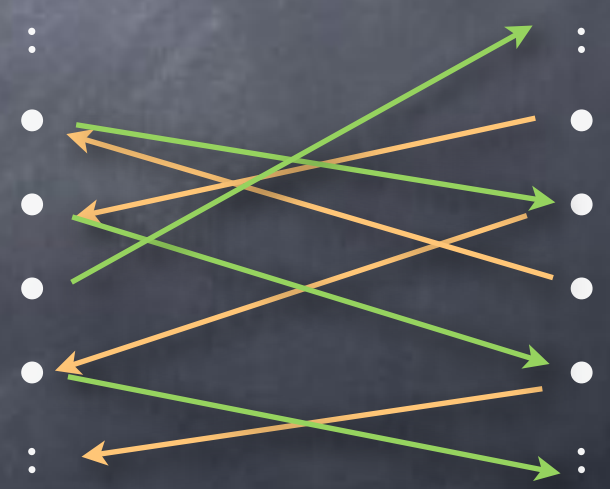

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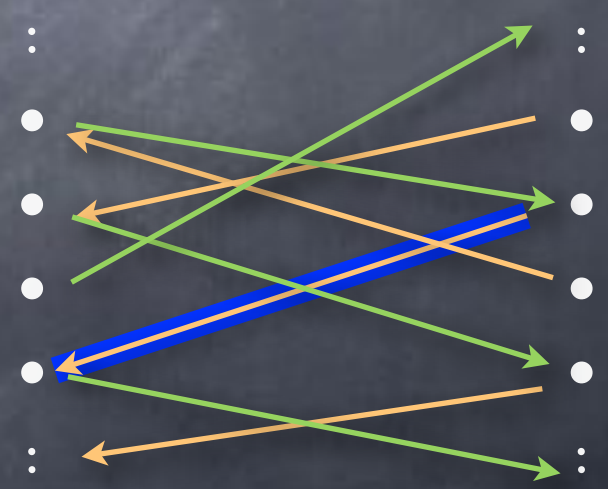

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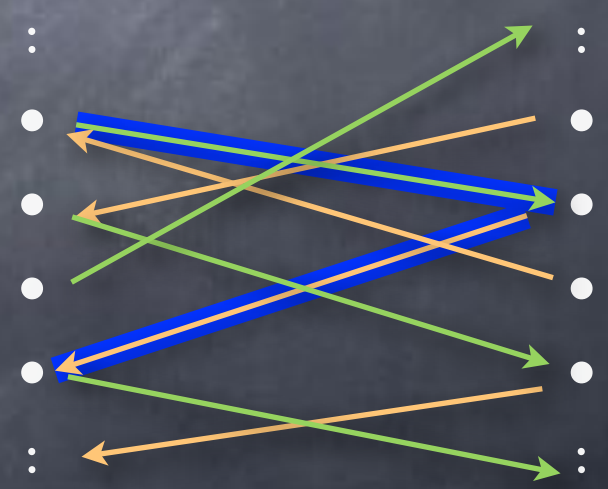

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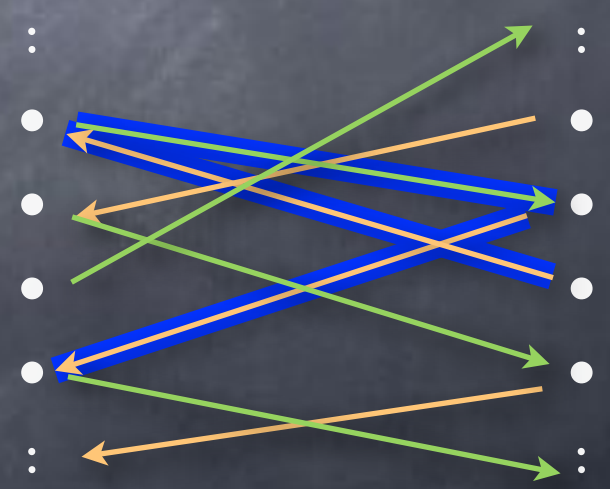

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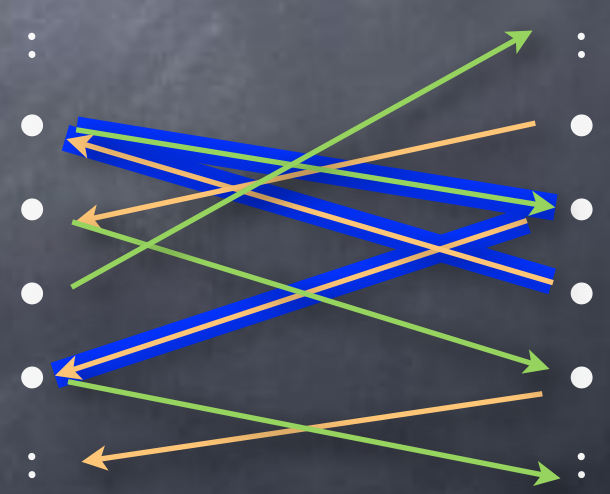

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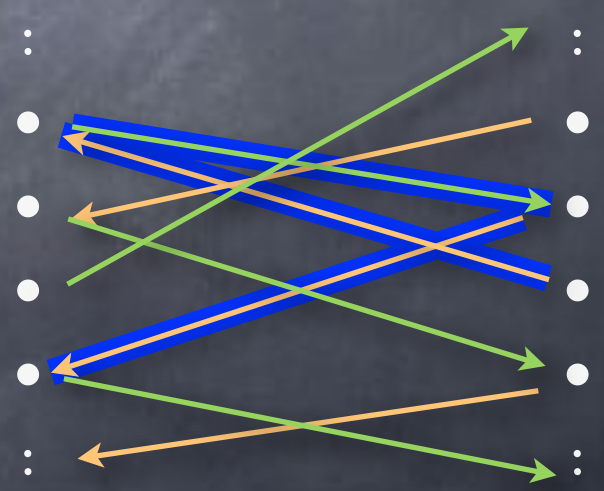

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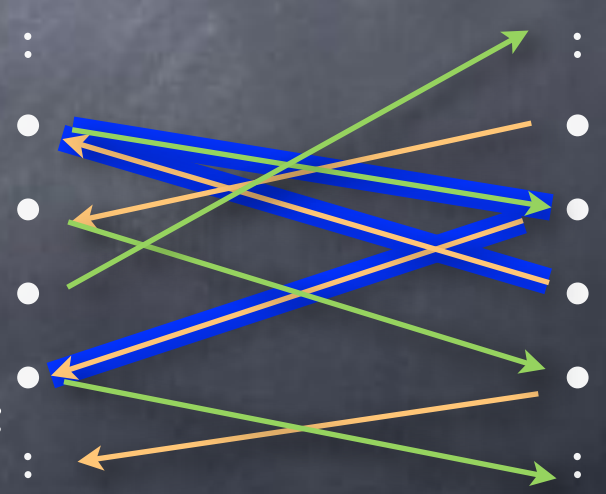

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- The set S of all finite-length strings made of [A-Z] is countably infinite

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  - Map an integer n to $A^n$ (string with n As). This is one-to-one.

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- Let $f:A \to B$ and $g:B \to C$ be two functions.
  Find the wrong statement

  A. if f and g are one-to-one, then $g \circ f$ is one-to-one
  B. if f and g are onto, then $g \circ f$ is onto
  C. if f is onto and $g \circ f$ is one-to-one, then f and g must both be one-to-one
  D. if $g \circ f$ is onto, then f and g are onto
  E. None of the above

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