Countability
Lecture 25
How do you count infinity?
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How do you make precise the intuition that there are more real numbers than integers? Both are infinite...
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How do you make precise the intuition that there are more real numbers than integers? Both are infinite...

When do we say two infinite sets A & B have the same size?
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How do you count infinity?

- How do you make precise the intuition that there are more real numbers than integers? Both are infinite...
- When do we say two infinite sets A & B have the same size?

**Definition:** \(|A| = |B|\) if there is a bijection from A to B.

Definition good for finite sets too.
How do you count infinity?

How do you make precise the intuition that there are more real numbers than integers? Both are infinite...

When do we say two infinite sets A & B have the same size?

**Definition:** \(|A| = |B|\) if there is a bijection from A to B.

\(|\mathbb{Z}| = |2\mathbb{Z}|. (2\mathbb{Z} = \text{evens}). f: \mathbb{Z} \rightarrow 2\mathbb{Z} \text{ defined as } f(x) = 2x \text{ is a bijection.} \)
How do you make precise the intuition that there are more real numbers than integers? Both are infinite...

When do we say two infinite sets A & B have the same size?

**Definition:** \( |A| = |B| \) if there is a bijection from A to B

\( |\mathbb{Z}| = |2\mathbb{Z}|. \) (\( 2\mathbb{Z} \) = evens). \( f: \mathbb{Z} \to 2\mathbb{Z} \) defined as \( f(x)=2x \) is a bijection

\( |\mathbb{Z}| = |\mathbb{N}|. \) bijection \( g: \mathbb{Z} \to \mathbb{N} : g(x) = 2x \) for \( x \geq 0, \) \( g(x)=2|x|-1 \) for \( x < 0 \)
How do you count infinity?

How do you make precise the intuition that there are more real numbers than integers? Both are infinite...

When do we say two infinite sets A & B have the same size?

**Definition:** $|A| = |B|$ if there is a bijection from A to B

- $|\mathbb{Z}| = |2\mathbb{Z}|$. ($2\mathbb{Z}$ = evens). $f: \mathbb{Z} \to 2\mathbb{Z}$ defined as $f(x) = 2x$ is a bijection

- $|\mathbb{Z}| = |\mathbb{N}|$. bijection $g: \mathbb{Z} \to \mathbb{N}$ : $g(x) = 2x$ for $x \geq 0$, $g(x) = 2|x|-1$ for $x < 0$

- $|\mathbb{N}| = |2\mathbb{Z}|$. $h: \mathbb{N} \to 2\mathbb{Z}$ defined as $h = f \circ g^{-1}$
Countable
A set $A$ is **countably infinite** if $|A| = |\mathbb{N}|$.
Countable

A set $A$ is \textit{countably infinite} if $|A|=|\mathbb{N}|$

i.e., there is a bijection between $A$ and $\mathbb{N}$
A set $A$ is **countably infinite** if $|A| = |\mathbb{N}|$

i.e., there is a bijection between $A$ and $\mathbb{N}$

Note: $|A| = |\mathbb{N}|$ iff $|A| = |\mathbb{Z}|$, $|A| = |\mathbb{Z}^2|$ etc.
A set $A$ is **countably infinite** if $|A| = |\mathbb{N}|$

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A set is **countable** if it is finite or countably infinite
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A set is **countable** if it is finite or countably infinite

Intuition: all “discrete” sets are countable
How do you count infinity?
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**We defined:** A is *countably infinite* if $|A| = |\mathbb{N}|$, i.e., if there is a bijection between $A$ and $\mathbb{N}$. 
How do you count infinity?

We defined: A is countably infinite if $|A| = |\mathbb{N}|$, i.e., if there is a bijection between A and $\mathbb{N}$.

$\mathbb{N}^2$ is countable. Bijection by ordering points in $\mathbb{N}^2$ on a "curve"

$(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), ...$

(i.e., $f(0)=(0,0)$, $f(1)=(1,0)$, $f(2)=(0,1)$ ...)
How do you count infinity?

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How do you count infinity?

We defined: A is countably infinite if |A| = |N|, i.e., if there is a bijection between A and N.

N² is countable. Bijection by ordering points in N² on a “curve”

(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), ...
(i.e., f(0)=(0,0), f(1)=(1,0), f(2)=(0,1) ...)

Note: (0,0), (1,0), (2,0), (3,0) ... will not give a bijection
How do you count infinity?

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\(\mathbb{Z}^2\) is countable. \(f: \mathbb{Z}^2 \rightarrow \mathbb{N}\) defined as \(f(a,b) = h (g(a),g(b))\), where \(g: \mathbb{Z} \rightarrow \mathbb{N}\) and \(h: \mathbb{N}^2 \rightarrow \mathbb{N}\) are bijections, is a bijection
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More generally, if A and B are countable, the \(A \times B\) is countable (extended to any finite number of sets by induction)
But Things Get Messy...
But Things Get Messy...

Is $\mathbb{Q}$ countable?
But Things Get Messy...

Is $|\mathbb{Q}|$ countable?

We saw bijection between $\mathbb{Z}^2$ and $\mathbb{N}$. Enough to find a bijection between $\mathbb{Q}$ and $\mathbb{Z}^2$. 
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Not immediately clear: not all pairs $(a,b)$ correspond to a distinct rational number $a/b$
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  - \(a\) and \(b\) can have a common divisor; also, trouble with \(b=0\)
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- But easier to construct a one-to-one function \(f: \mathbb{Q} \rightarrow \mathbb{Z}^2\) as \(f(x) = (p,q)\) where \(x=p/q\) is the “canonical representation” of \(x\) (i.e., \(gcd(p,q)=1\) and \(q > 0\)).
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  - Hence one-to-one function \(g \circ f: \mathbb{Q} \to \mathbb{N}\), where \(g: \mathbb{Z}^2 \to \mathbb{N}\) is a bijection.
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- Hence one-to-one function $g \circ f: \mathbb{Q} \rightarrow \mathbb{N}$, where $g: \mathbb{Z}^2 \rightarrow \mathbb{N}$ is a bijection

- Also can construct a one-to-one function $h: \mathbb{N} \rightarrow \mathbb{Q}$ as $h(a)=a$
But Things Get Messy...
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Is $\mathbb{Q}$ countable?
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Is $\mathbb{Q}$ countable?

One-to-one functions $f_1: \mathbb{Q} \rightarrow \mathbb{N}$ and $f_2: \mathbb{N} \rightarrow \mathbb{Q}$
But Things Get Messy...

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Intuitively, if a one-to-one function from $A$ to $B$, $|A| \leq |B|$
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**Definition:** \(|A| \leq |B|\) if there is a one-to-one function from \(A\) to \(B\)
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- So $|\mathbb{Q}| \leq |\mathbb{N}|$ and $|\mathbb{N}| \leq |\mathbb{Q}|$
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**Definition:** $|A| \leq |B|$ if there is a one-to-one function from $A$ to $B$

So $|\mathbb{Q}| \leq |\mathbb{N}|$ and $|\mathbb{N}| \leq |\mathbb{Q}|$

Want to show $|\mathbb{Q}| = |\mathbb{N}|$ (i.e., a bijection between $\mathbb{Q}$ and $\mathbb{N}$)
Bijection from Two Injections
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**Theorem [CBS]:** There is a bijection from A to B if and only if there is a one-to-one function from A to B, and a one-to-one function from B to A
Bijection from Two Injections

**Theorem [CBS]:** There is a bijection from $A$ to $B$ if and only if there is a one-to-one function from $A$ to $B$, and a one-to-one function from $B$ to $A$

Restated: $|A| = |B| \iff |A| \leq |B| \text{ and } |B| \leq |A|$
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This statement is trivial for finite sets.
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Proof idea: Let $f:A \rightarrow B$ and $g:B \rightarrow A$ (one-to-one). From any given $a \in A$, consider a chain obtained by following the arrows backwards

Trivial for finite sets
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\[
\begin{align*}
\vdots & \quad \vdots \\
\vdots & \quad \vdots \\
\vdots & \quad \vdots \\
\vdots & \quad \vdots \\
\vdots & \quad \vdots \\
\vdots & \quad \vdots \\
\vdots & \quad \vdots \\
\end{align*}
\]

- Trivial for finite sets

Thursday, December 6, 12
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Theorem [CBS]: There is a bijection from \( A \) to \( B \) if and only if there is a one-to-one function from \( A \) to \( B \), and a one-to-one function from \( B \) to \( A \).

Restated: \(|A| = |B| \iff |A| \leq |B| \text{ and } |B| \leq |A|\)

Proof idea: Let \( f: A \rightarrow B \) and \( g: B \rightarrow A \) (one-to-one). From any given \( a \in A \), consider a chain obtained by following the arrows backwards.

One-to-one \( \Rightarrow \) no forking in a chain.

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Proof idea: Let \(f: A \to B\) and \(g: B \to A\) (one-to-one). From any given \(a \in A\), consider a chain obtained by following the arrows backwards.

- One-to-one \(\Rightarrow\) no forking in a chain
- Chain could end in an A node, end in a B node or go on forever, possibly looping (types A, B and C)

\(\triangleright\text{Trivial for finite sets}\)
**Bijection from Two Injections**

**Theorem [CBS]:** There is a bijection from $A$ to $B$ if and only if there is a one-to-one function from $A$ to $B$, and a one-to-one function from $B$ to $A$.

Restated: $|A| = |B| \iff |A| \leq |B|$ and $|B| \leq |A|$

Proof idea: Let $f: A \to B$ and $g: B \to A$ (one-to-one). From any given $a \in A$, consider a chain obtained by following the arrows backwards.

- **One-to-one** ⇒ no forking in a chain
- Chain could end in an A node, end in a B node or go on forever, possibly looping (types A, B and C)
- Let $h: A \to B$ s.t. $h(a) = f(a)$ if $a$'s chain type A; else $h(a) = b$ s.t. $g(b) = a$.

Trivial for finite sets
Bijection from Two Injections
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Since \(|\mathbb{Q}| \leq |\mathbb{N}|\) and \(|\mathbb{N}| \leq |\mathbb{Q}|\), by CBS-theorem \(|\mathbb{Q}| = |\mathbb{N}|\)
Bijection from Two Injections

Since \(|\mathbb{Q}| \leq |\mathbb{N}|\) and \(|\mathbb{N}| \leq |\mathbb{Q}|\), by CBS-theorem \(|\mathbb{Q}| = |\mathbb{N}|\)

\(\mathbb{Q}\) is countable
Bijection from Two Injections

- Since $|\mathbb{Q}| \leq |\mathbb{N}|$ and $|\mathbb{N}| \leq |\mathbb{Q}|$, by CBS-theorem $|\mathbb{Q}| = |\mathbb{N}|$

- $\mathbb{Q}$ is countable

- The set $S$ of all finite-length strings made of [A–Z] is countably infinite
Bijection from Two Injections

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- The set $S$ of all finite-length strings made of [A-Z] is countably infinite

- Interpret A to Z as the non-zero digits in base 27. Given $s \in S$, interpret it as a number. This mapping $(S \rightarrow \mathbb{N})$ is one-to-one
Bijection from Two Injections

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- Interpret A to Z as the non-zero digits in base 27. Given $s \in S$, interpret it as a number. This mapping $(S \rightarrow \mathbb{N})$ is one-to-one

- Map an integer $n$ to $A^n$ (string with $n$ As). This is one-to-one.
Question
Let \( f: A \to B \) and \( g: B \to C \) be two functions.

Find the wrong statement

A. if \( f \) and \( g \) are one-to-one, then \( g \circ f \) is one-to-one
B. if \( f \) and \( g \) are onto, then \( g \circ f \) is onto
C. if \( f \) is onto and \( g \circ f \) is one-to-one, then \( f \) and \( g \) must both be one-to-one
D. if \( g \circ f \) is onto, then \( f \) and \( g \) are onto
E. None of the above
The Uncountable
The Uncountable

Claim: \( \mathbb{R} \) is uncountable
The Uncountable

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Related claims:
The Uncountable

- **Claim:** $\mathbb{R}$ is uncountable
- **Related claims:**
  - Set $S$ of all infinitely long binary strings is uncountable
The Uncountable

Claim: \( \mathbb{R} \) is uncountable

Related claims:

- Set \( S \) of all infinitely long binary strings is uncountable
  - Contrast with set of all finitely long binary strings, which is a countably infinite set
The Uncountable

Claim: $\mathbb{R}$ is uncountable

Related claims:
- Set $S$ of all infinitely long binary strings is uncountable
- Contrast with set of all finitely long binary strings, which is a countably infinite set
- The power-set of $\mathbb{N}$, $\mathcal{P}(\mathbb{N})$ is uncountable
The Uncountable

- **Claim:** \( \mathbb{R} \) is uncountable

- Related claims:
  - Set \( S \) of all *infinitely long* binary strings is uncountable
    - Contrast with set of all *finitely long* binary strings, which is a countably infinite set
  - The power-set of \( \mathbb{N} \), \( \mathcal{P}(\mathbb{N}) \) is uncountable

- There is a bijection \( f: S \to \mathcal{P}(\mathbb{N}) \) defined as \( f(s) = \{ i \mid s_i = 1 \} \)
The Uncountable

- **Claim:** $\mathbb{R}$ is uncountable

- **Related claims:**
  - Set $S$ of all infinitely long binary strings is uncountable
  - Contrast with set of all finitely long binary strings, which is a countably infinite set
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There is a bijection \( f: S \rightarrow \mathcal{P}(\mathbb{N}) \) defined as \( f(s) = \{ i | s_i = 1 \} \)

e.g., set of even numbers corresponds to the string 101010...
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- **Related claims:**
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  - There is a bijection \( f: S \rightarrow \mathcal{P}(\mathbb{N}) \) defined as \( f(s) = \{ i \mid s_i = 1 \} \)

- **How do we show something is not countable?!**

  e.g., set of even numbers corresponds to the string 101010...
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  - The power-set of \( \mathbb{N} \), \( \mathcal{P}(\mathbb{N}) \) is uncountable
  - There is a bijection \( f: S \rightarrow \mathcal{P}(\mathbb{N}) \) defined as \( f(s) = \{ i \mid s_i = 1 \} \)

- How do we show something is not countable?!
- Cantor’s “diagonal slash”

---

e.g., set of even numbers corresponds to the string 101010...
Cantor's Diagonal Slash
Cantor’s Diagonal Slash

Take any function $f: \mathbb{N} \to \mathcal{P}(\mathbb{N})$
Cantor's Diagonal Slash

- Take any function f: $\mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})$
- Make a binary table with $T_{ij} = 1$ iff $j \in f(i)$

<table>
<thead>
<tr>
<th></th>
<th>f(0)</th>
<th>f(1)</th>
<th>f(2)</th>
<th>f(3)</th>
<th>f(4)</th>
<th>f(5)</th>
<th>f(6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(0) =</td>
<td>1 0 0 1 0 0 0 0</td>
<td>0 0 1 0 1 0 0</td>
<td>1 1 1 1 1 1 1 1</td>
<td>1 1 0 1 0 1 0</td>
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<td>f(1) =</td>
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<td>f(2) =</td>
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<tr>
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<td>f(4) =</td>
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<td>f(5) =</td>
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<td>f(6) =</td>
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Cantor’s Diagonal Slash

Take any function $f: \mathbb{N} \to \mathcal{P}(\mathbb{N})$

Make a binary table with $T_{ij} = 1$ iff $j \in f(i)$

Consider the set $X \subseteq \mathbb{N}$ corresponding to the “flipped diagonal”

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Cantor’s Diagonal Slash

- Take any function \( f : \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N}) \)
- Make a binary table with \( T_{ij} = 1 \) iff \( j \in f(i) \)
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\begin{verbatim}
f(0) = 1 0 0 1 0 0 0 0
f(1) = 0 0 1 0 1 0 0 0
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f(3) = 1 1 0 1 0 1 0 0
f(4) = 1 1 0 0 0 0 1 0
f(5) = 0 0 0 0 0 0 0 1
f(6) = 0 1 0 1 0 1 0 0
\end{verbatim}
Cantor’s Diagonal Slash

- Take any function $f: \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})$
- Make a binary table with $T_{ij} = 1$ iff $j \in f(i)$
- Consider the set $X \subseteq \mathbb{N}$ corresponding to the “flipped diagonal”
  - $X = \{ j \mid T_{jj} = 0 \}$
  - $= \{ j \mid j \notin f(j) \}$

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Cantor's Diagonal Slash

- Take any function $f : \mathbb{N} \rightarrow \mathcal{P} (\mathbb{N})$
- Make a binary table with $T_{ij} = 1$ iff $j \in f(i)$
- Consider the set $X \subseteq \mathbb{N}$ corresponding to the "flipped diagonal"
  - $X = \{ j \mid T_{jj} = 0 \}$
  - $= \{ j \mid j \notin f(j) \}$
- $X$ doesn't appear as a row in this table (why?)

| $f(0)$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| $f(1)$ | 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| $f(2)$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $f(3)$ | 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| $f(4)$ | 1 | 1 | 0 | 0 | 0 | 0 | 1 |
| $f(5)$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| $f(6)$ | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
Cantor's Diagonal Slash

- Take any function \( f: \mathbb{N} \to \mathcal{P}(\mathbb{N}) \)
- Make a binary table with \( T_{ij} = 1 \) iff \( j \in f(i) \)
- Consider the set \( X \subseteq \mathbb{N} \)
  - corresponding to the “flipped diagonal”
  - \( X = \{ j \mid T_{jj} = 0 \} = \{ j \mid j \notin f(j) \} \)
- \( X \) doesn’t appear as a row in this table (why?)
- So \( f \) not onto