Mathematical Induction

Proof by Programming

Lecture 14

Proof by Induction

Proof by Induction

- To prove $\forall n \in \mathbb{Z}^+ \; P(n)$:

Proof by Induction

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	⋮

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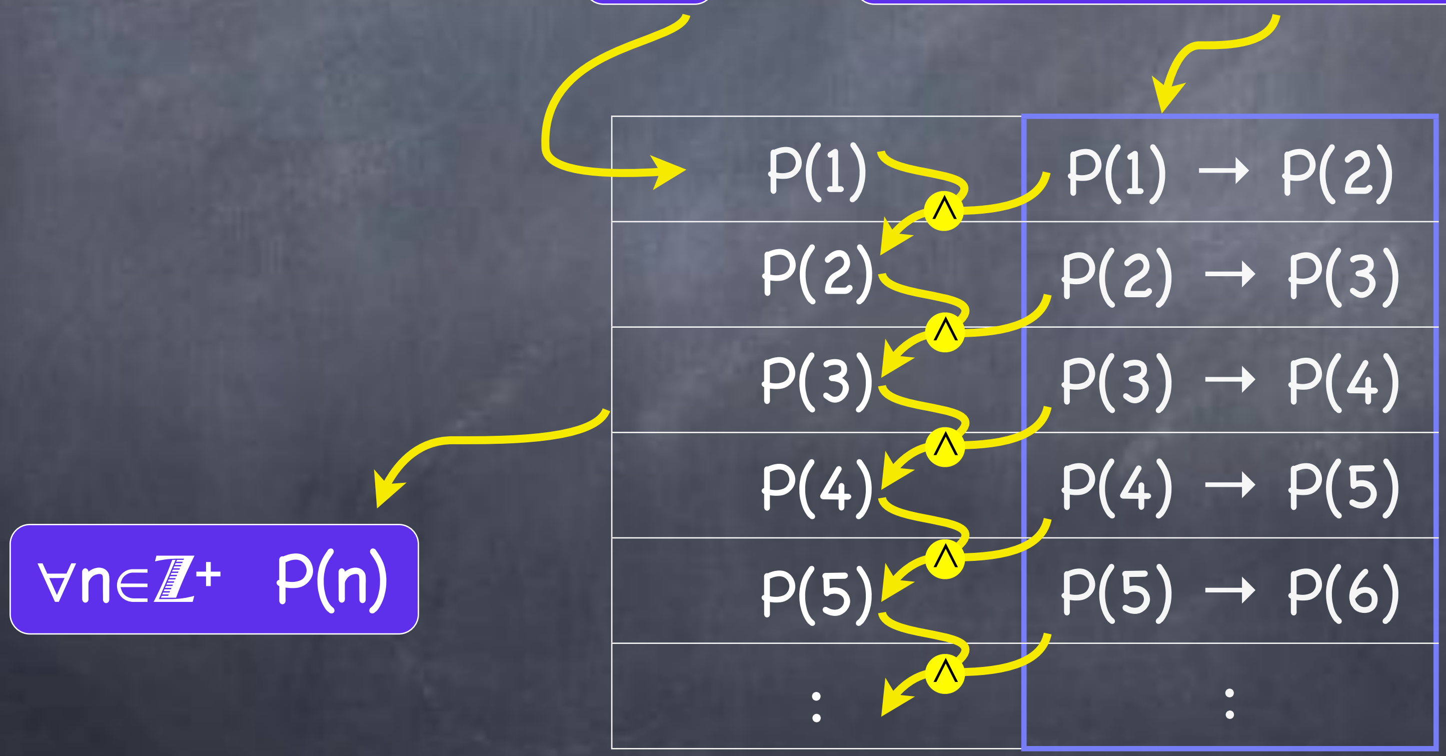

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Mathematical Induction

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Proof by Induction

- To prove $\forall n \in \mathbb{Z}^+ \ P(n)$:

  Base case

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Proof by Induction

- To prove $\forall n \in \mathbb{Z}^+ \;\; P(n)$:

  Base case

  Induction hypothesis

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Base case

Induction step

Induction hypothesis

Proof by Induction

- To prove $\forall n \in \mathbb{Z}^+ \ \ P(n)$:
  - First, we prove $\underbrace{P(1)}_{\text{Base case}}$ and $\underbrace{\forall k \in \mathbb{Z}^+ \ \ \underbrace{P(k)}_{\text{Induction hypothesis}} \rightarrow \underbrace{P(k+1)}_{\text{To prove}}}_{\text{Induction step}}$
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  - Induction step: For all integers $k \geq 0$
    Induction hypothesis: True for n=k. i.e., $k^3-k = 3m$
    To prove: True for n=k+1. i.e., $3 \mid (k+1)^3-(k+1)$

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    - $$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= (k^3 - k) + 3k^2+3k \\ &= 3m + 3k^2 + 3k \quad ✔\end{aligned}$$

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- The non-inductive proof: $n^3-n = n(n^2-1) = (n-1)n(n+1)$.
  $3|n(n+1)(n+2)$ since one of n, $(n+1)$, $(n+2)$ is $\equiv 0 \pmod 3$

Tromino Tiling

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- L-trominoes can be used to tile a "punctured" $2^n \times 2^n$ grid (punctured = one cell removed), for all positive integers n

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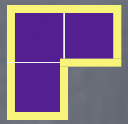

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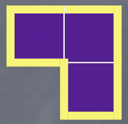

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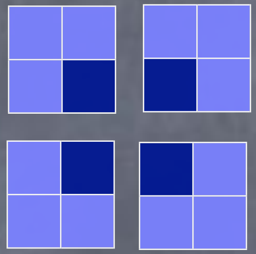

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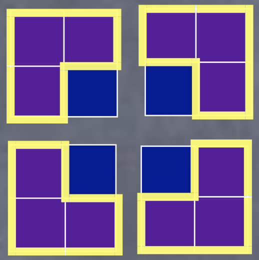

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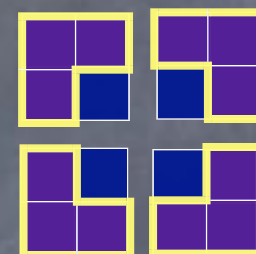

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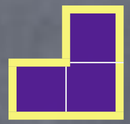

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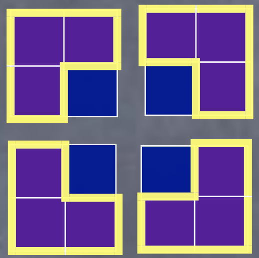

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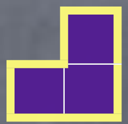

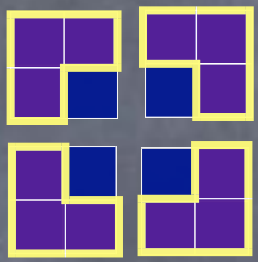

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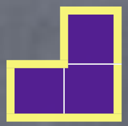

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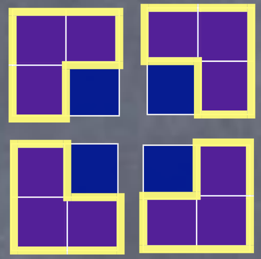

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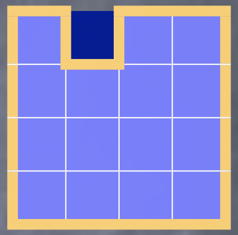

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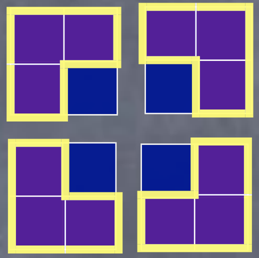

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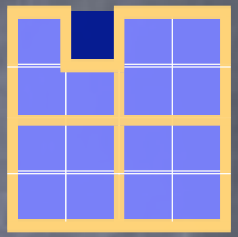

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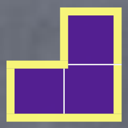

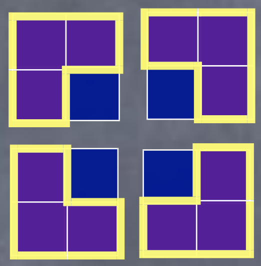

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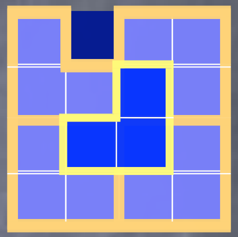

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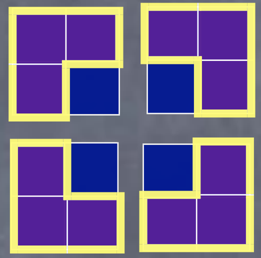

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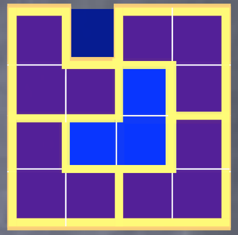

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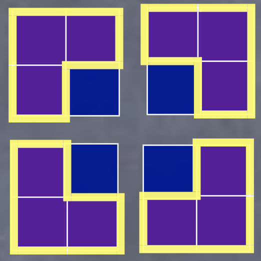

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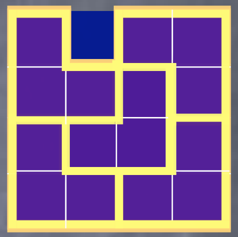

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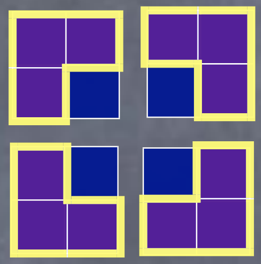

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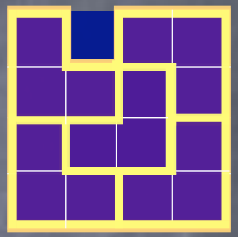

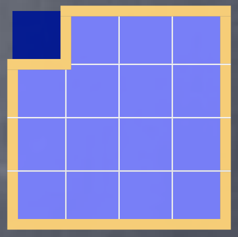

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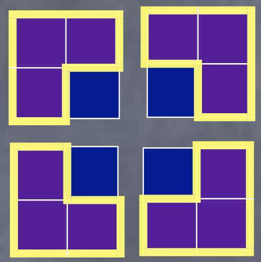

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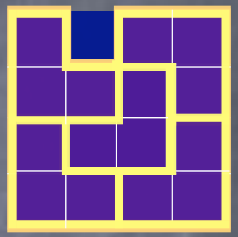

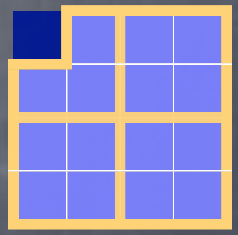

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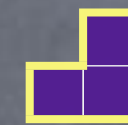

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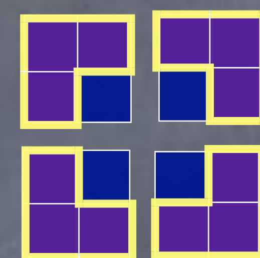

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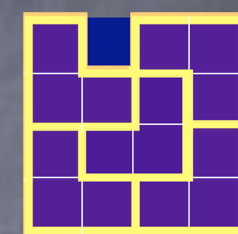

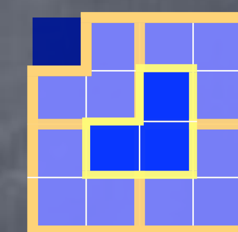

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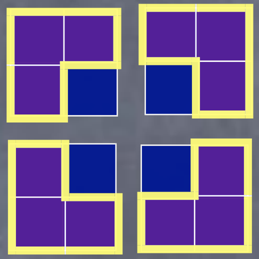

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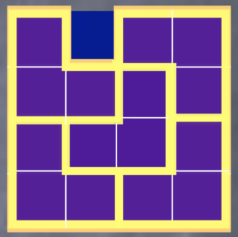

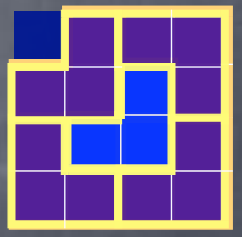

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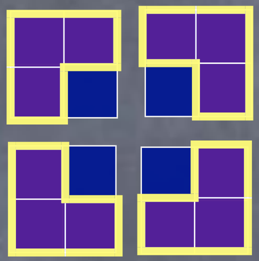

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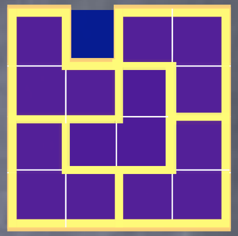

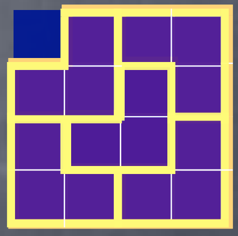

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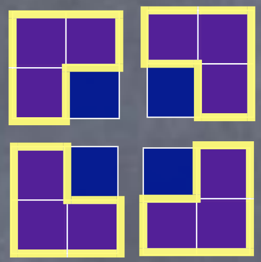

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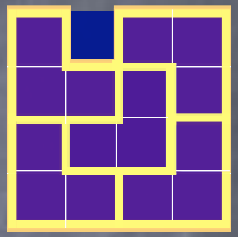

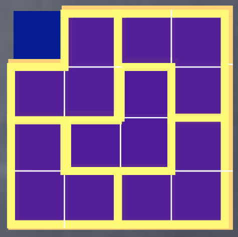

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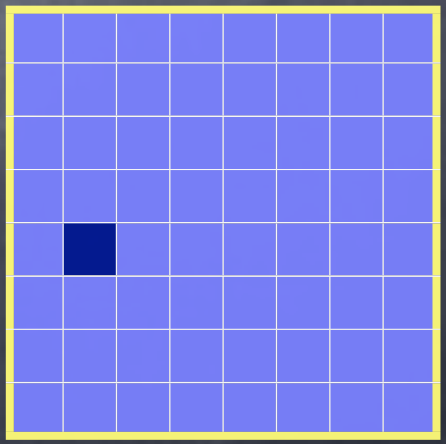

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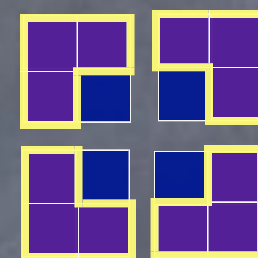

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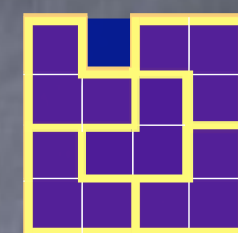

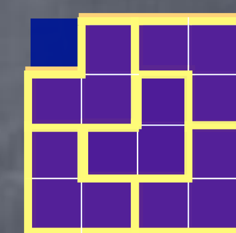

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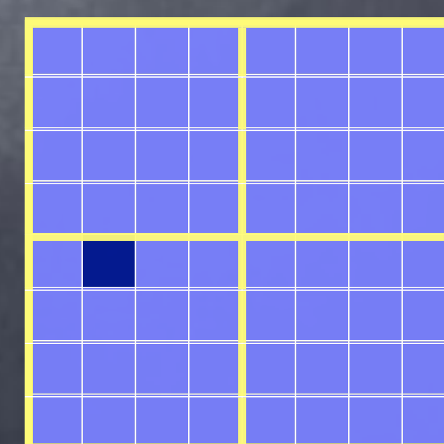

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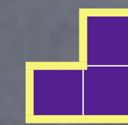

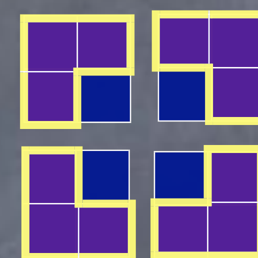

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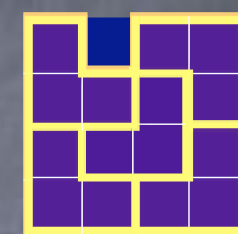

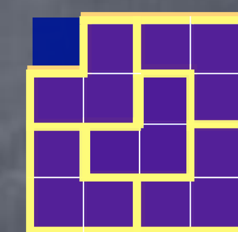

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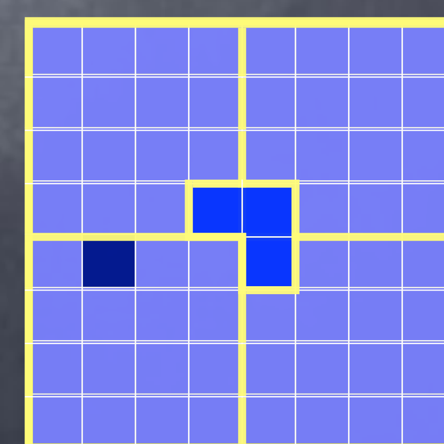

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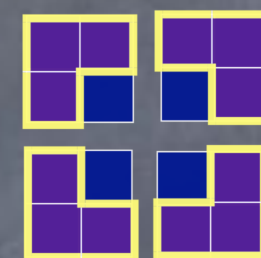

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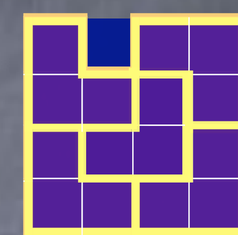

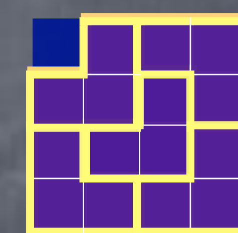

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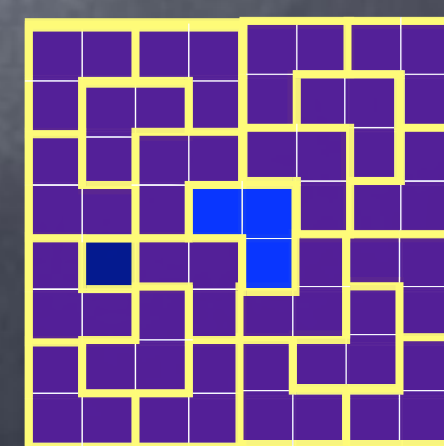

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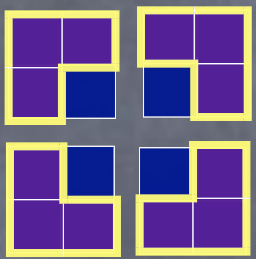

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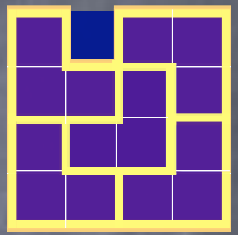

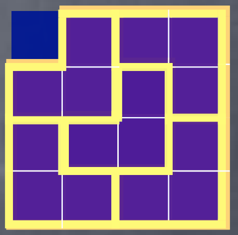

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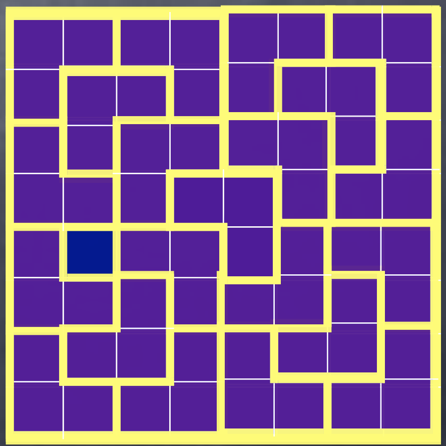

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Proof describes a recursive algorithm for coloring with $\Delta(G)+1$ colors

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- What is "the (most) correct" induction step?

A. For all $k \geq 3$, C_k not bi-partite → C_{k+1} is not bi-partite
B. For all $k \geq 1$, C_{2k-1} not bi-partite → C_{2k+1} is not bi-partite
C. For all $k \geq 1$, C_{2k+1} not bi-partite → C_{2k-1} is not bi-partite
D. For all $k \geq 2$, C_{2k+1} not bi-partite → C_{2k+3} is not bi-partite
E. For all $k \geq 2$, C_{2k-1} not bi-partite → C_{2k+1} is not bi-partite

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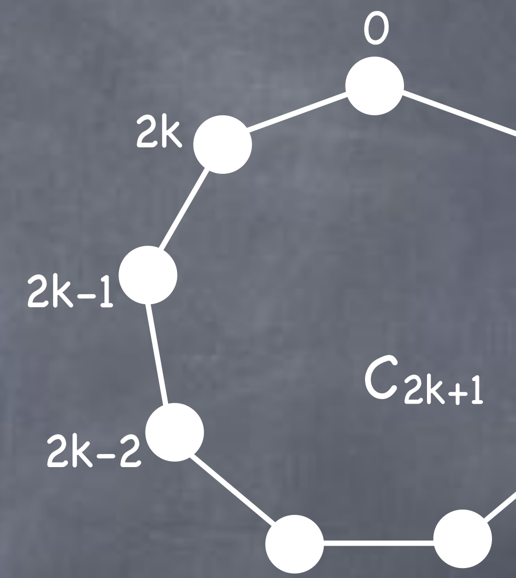

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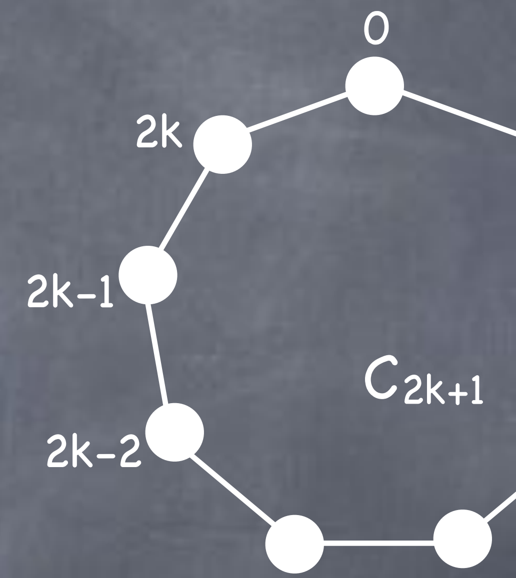

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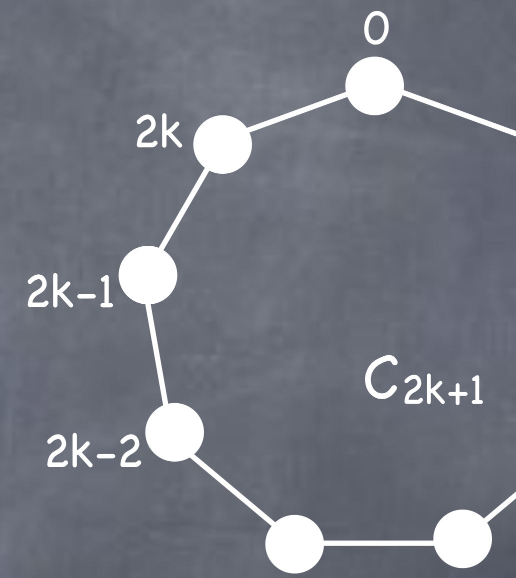

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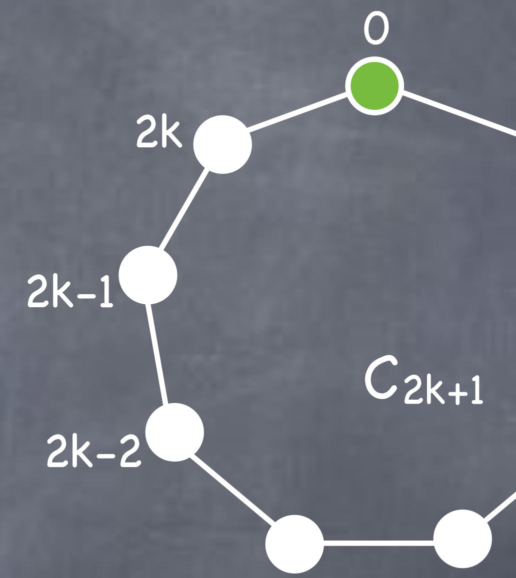

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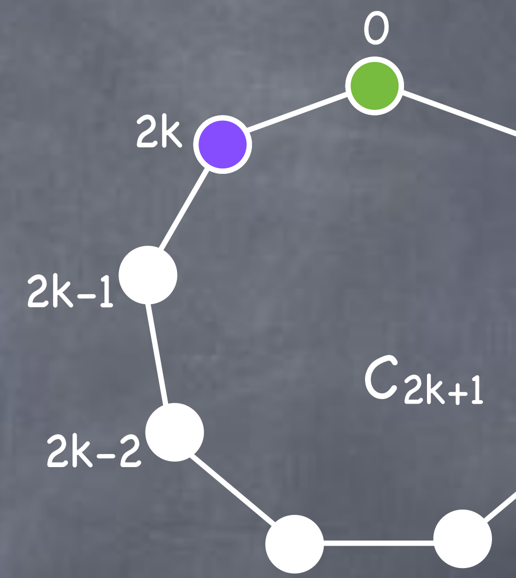

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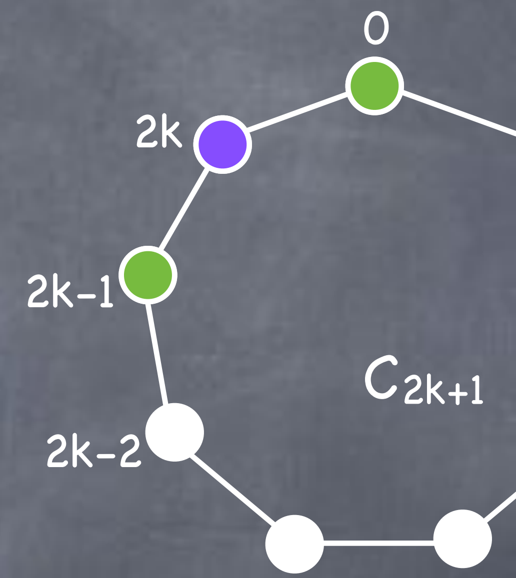

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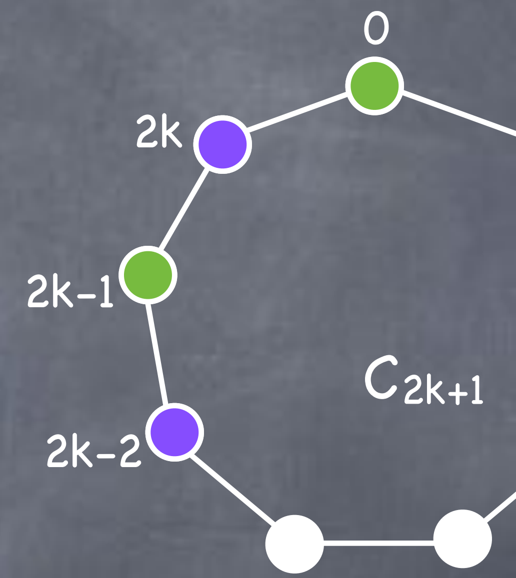

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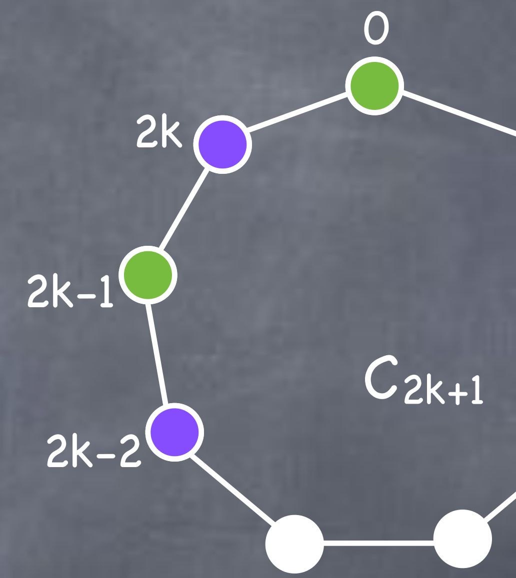

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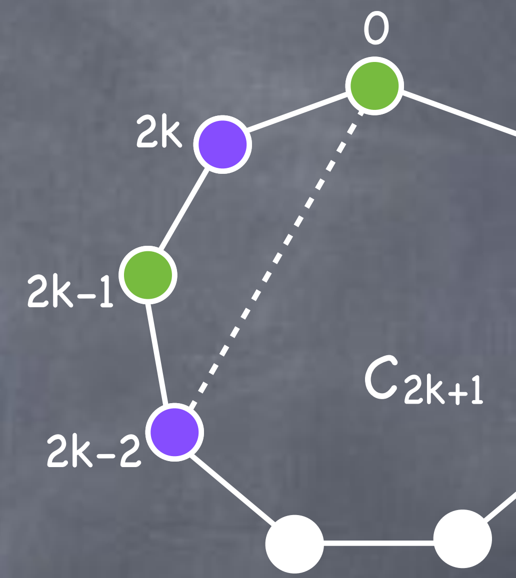

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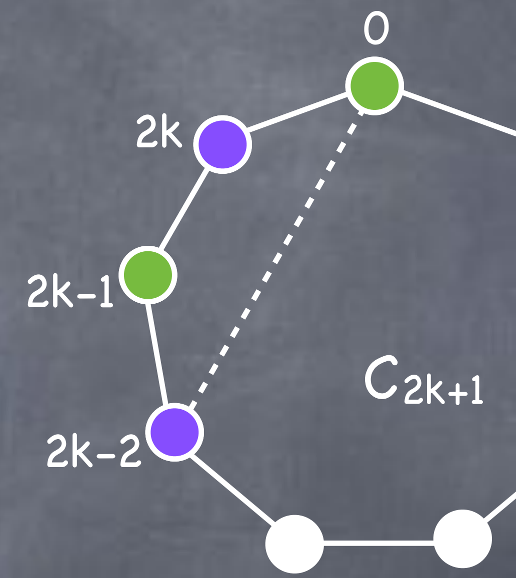

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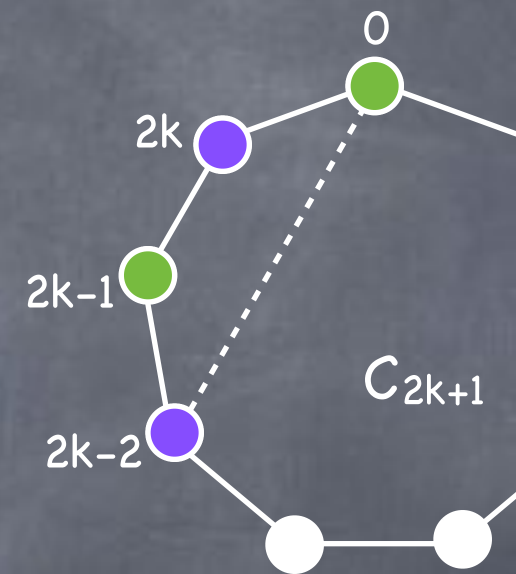

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  - So c respects all edges of C_{2k-1}.
  - So $c':\{0,..,2k-2\} \rightarrow \{1,2\}$ with $c'(u)=c(u)$ is a valid coloring of C_{2k-1}.

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The fact that for any n, we can run this procedure to generate a proof for P(n), and hence for any n, P(n) holds.

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Could be thought of as weak induction with $Q(n) \triangleq \forall m \in [1,n] \ \ P(m)$

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Prime Factorization

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Need some more work to show <u>unique</u> factorization.

<u>p prime $\wedge$ $p|ab$ $\rightarrow$ $p|a \vee p|b$</u>

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  - So $k = 4(a+1) + 5b$.

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  - Case 1: Alice removes all k+1 from one pile. Then Bob wins.

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  - Case 1: Alice removes all $k+1$ from one pile. Then Bob wins.
  - Case 2: Alice removes j, $1 \leq j \leq k$ from one pile. After Bob's move $k+1-j$ left in each pile. By induction hypothesis, Bob will always win from here.