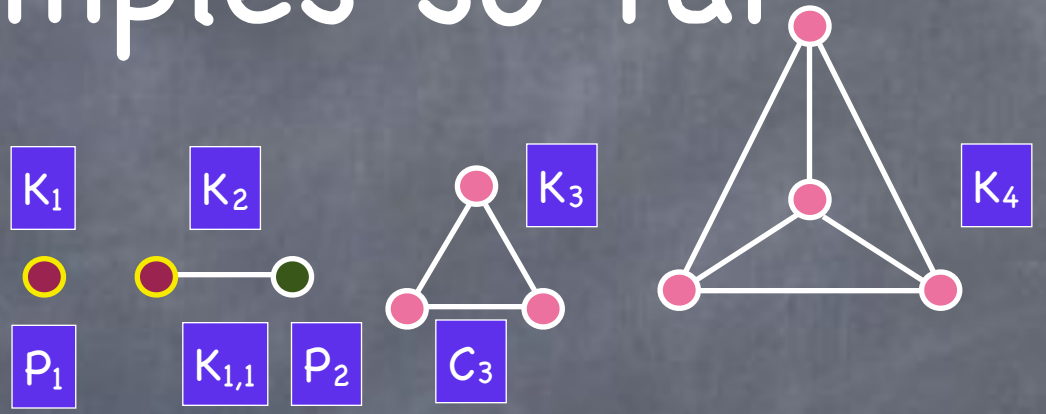
Graphs

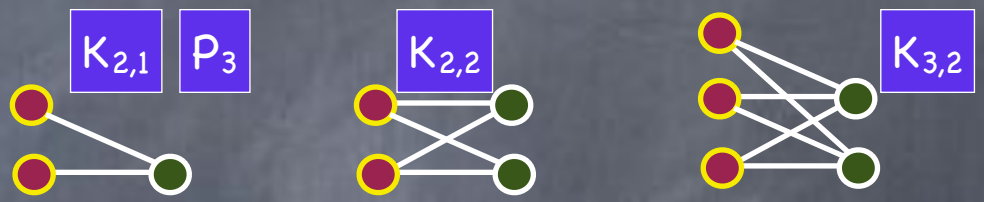
Lecture 13

Examples so far

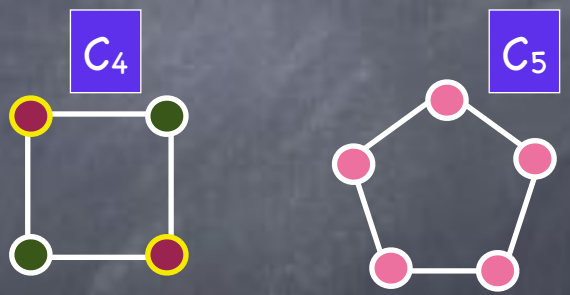
- Complete graph K_n



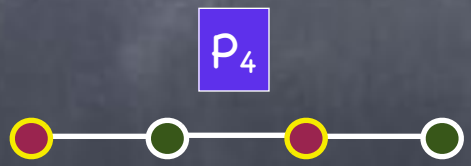
- Complete bi-partite graph $K_{m,n}$



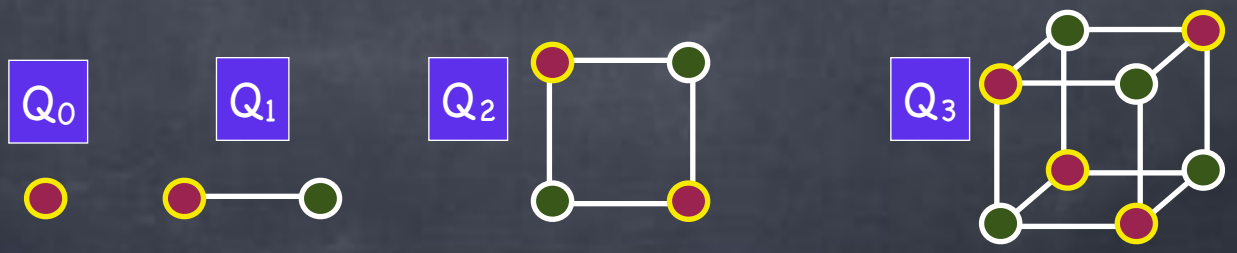
- Cycle graph C_n



- Path graph P_n



- Hypercube graph Q_n



Graph Coloring

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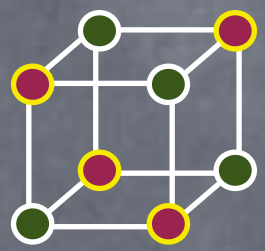

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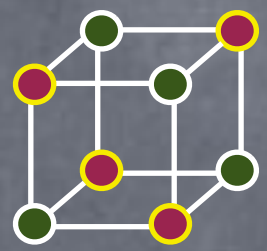

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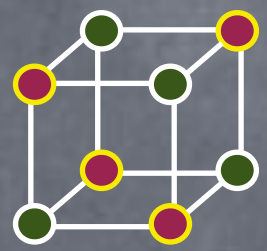

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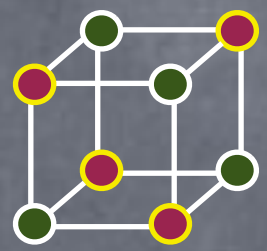

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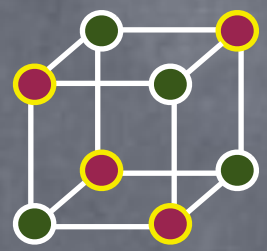

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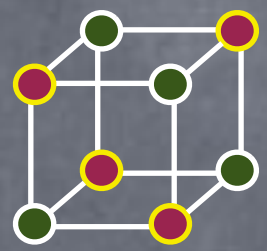

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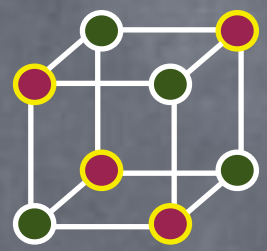

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- Summary: One way to show $k_{lower} \leq \chi(G) \leq k_{upper}$
  Show a coloring $c:V \rightarrow \{1,\ldots,k_{upper}\}$
  And show a subgraph H with $k_{lower} \leq \chi(H)$

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  - $\rightarrow$: We will prove the contrapositive: i.e., that if G not isomorphic to $K_{|V|}$, then $\chi(G) \neq |V|$.
    - Suppose G not isomorphic to $K_{|V|}$. So G should have at least two distinct nodes u, v s.t. $\{u,v\} \notin E$. Consider the coloring which assigns colors $\{1,..., |V|-2\}$ to the nodes in $V-\{u,v\}$ and the color $|V|-1$ to both u and v. This is a valid coloring (because $f(x)=f(y) \rightarrow \{x,y\} \notin E$). So $\chi(G) \leq |V|-1$

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  - But computing chromatic number in general is believed to be "hard" (known to be in a class of problems called "NP-hard")

Graphs in action

Graph Coloring in Action

- Many problems can be modeled as a graph coloring problem
- Resource scheduling: allocate "**resources**" (e.g. time slots, radio frequencies) to "**demands**" (exams, radio stations). Use as few resources as possible. Same resource can be used to satisfy multiple demands, as long as they don't have a "**conflict**" (same student, inhabited area with signal overlap).
  - Create a "conflict graph": Demands are the nodes; connect them by an edge if they have a conflict
  - Color the graph with as few colors as possible
  - Allocate one resource per color

Shortest Paths in Action

- Obvious example: nodes correspond to locations on a map and edges are roads, optic fibers etc.
  - Weighted edges: each edge has its own "length" (instead of 1)
- But also over more abstract graphs
  - e.g., Graph-based models in AI/machine-learning for modeling probabilistic systems
    - e.g., a graph, modeling speech production: nodes correspond to various "states" the vocal chords/lips etc. could be in while producing a given a sound sequence. Edges show transitions (next state) over time. Shortest path in this graph gives the "most likely" word that was spoken.

Graphs in action

Network Flow

- Transporting (people, material, data) over a network of links with limited capacity, and possibly cost for bandwidth use
  - A central problem in "Operations Research"
- (Bipartite) Matching problem: pair up every node with one of its neighbors, so that no node is left alone or has more than one partner
- "Max-flow Min-cut" theorem: can route as much flow from one node to another, as the smallest "cut" permits
  - Efficient algorithms known
  - More challenging when there are multiple flows to be routed together on the same network

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- Design graphs with **low degree** (look at a few (neighboring) pieces of data at a time; reduce hardware cost), but **good "connectivity"** -- i.e., (possibly many) short paths between any two nodes (to reach the required piece of data quickly, by taking a path over the graph; to route data quickly)

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- Very efficient algorithms known for relevant graph problems
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- But many other graph problems are known to be "NP-hard"
  - e.g., Traveling Salesperson Problem (TSP): visit all cities, by traveling the least distance

Mathematical Induction

Proof by Programming

Lecture 13

Programming a Proof

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    - Describe a procedure that can generate the proof for each n

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  - But what if you don't have one?
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  - But what if you don't have one?
  - Get it from the ATM by feeding it a $98 bill. And if you don't have a $98 bill, get that from the ATM... Enough to start with a $1 bill!
- To get a $100 bill, you need two things: some smaller bill ($1 would do) and the funny ATM

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A Proof in Two Acts

Proving the Base Case

- Let $f(n) = \Sigma_{(i=1 \text{ to } n)}\ i^2$ and $g(n) = n(n+1)(2n+1)/6$
- $\forall n \in \mathbb{Z}^+,\ \ f(n) = g(n)$
  - $f(1) = 1,\ \ g(1) = 1$ ✔ (that is our $1 bill)
- What is the funny ATM?
  - $\forall n \in \mathbb{Z}^+,\ \ (\underline{f(n) = g(n)}) \rightarrow (\underline{f(n+1) = g(n+1)})$
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Proving the Base Case

Proving original statement: setting up a press to print any \$n bill.

ATM needs to only change the value printed on the bill

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  - $\forall n \in \mathbb{Z}^+,\ (f(n) = g(n)) \rightarrow (f(n+1) = g(n+1))$
  - We need to build this ATM: i.e., prove this statement
    - This is easier than proving the original statement
  - We also need to procure a \$1 bill (we already did by proving f(1)=g(1))

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Building the ATM

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The Induction Step

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- $f(k+1) = f(k) + (k+1)^2$
  $= g(k) + (k+1)^2$ By induction hypothesis
  $= k(k+1)(2n+1)/6 + (k+1)^2$

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- Now some algebraic manipulation:

  $f(k+1) = k(k+1)(2k+1)/6 + (k+1)^2 = (k+1)\,[\,2k^2 + k + 6k + 6\,]/6$

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  $g(k+1) = (k+1)(k+2)(2(k+1)+1)/6 = (k+1)\ [\ (k+2)(2k+3)\ ]/6$
- $f(k+1) = g(k+1)$

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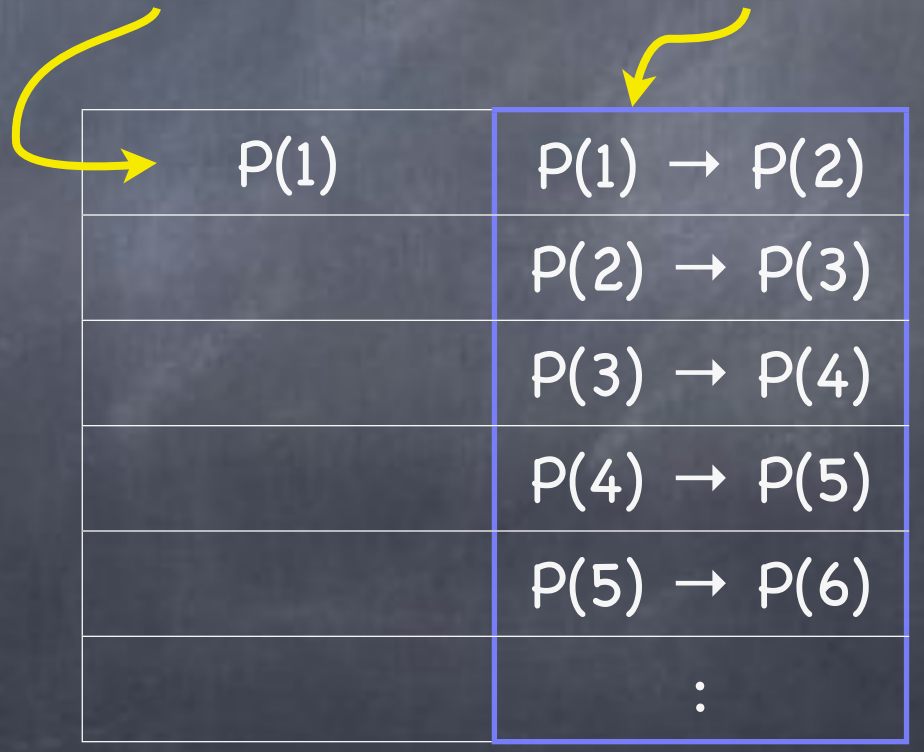

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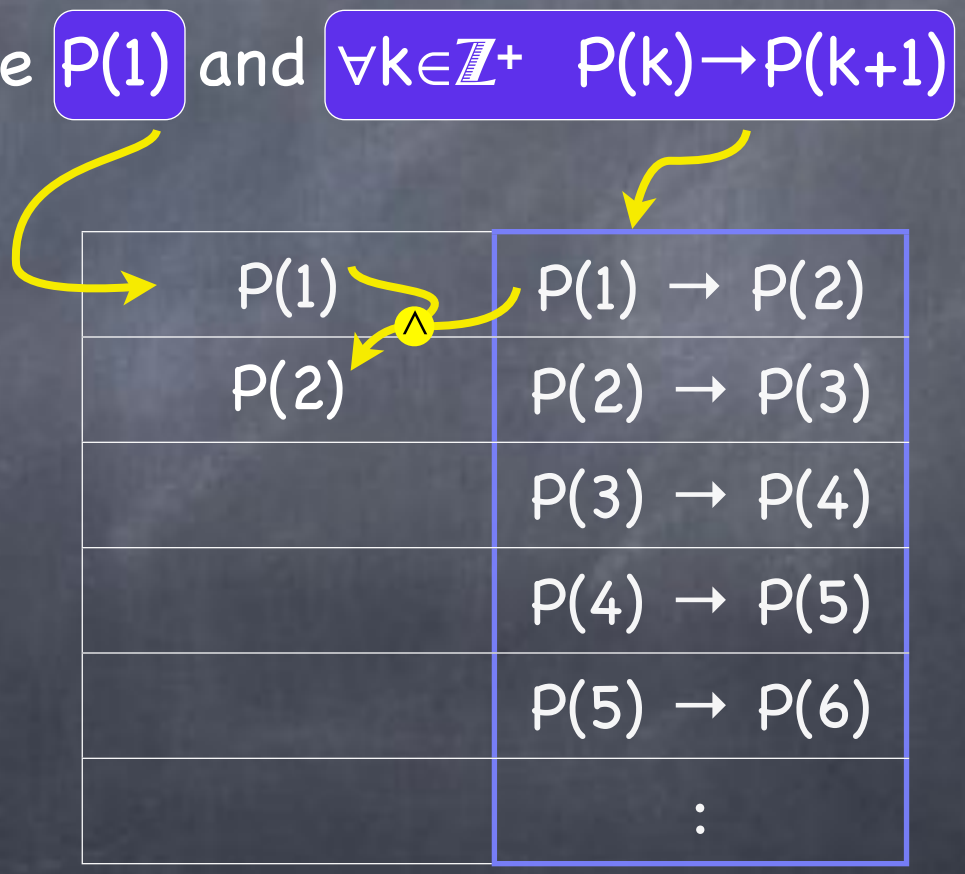

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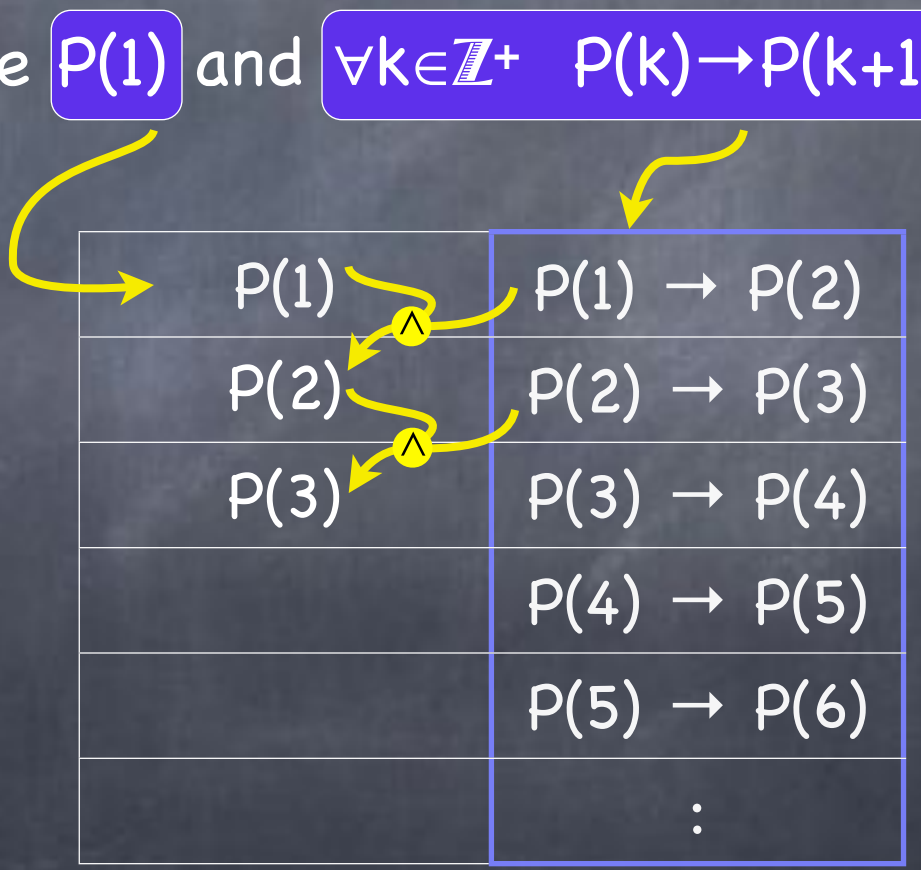

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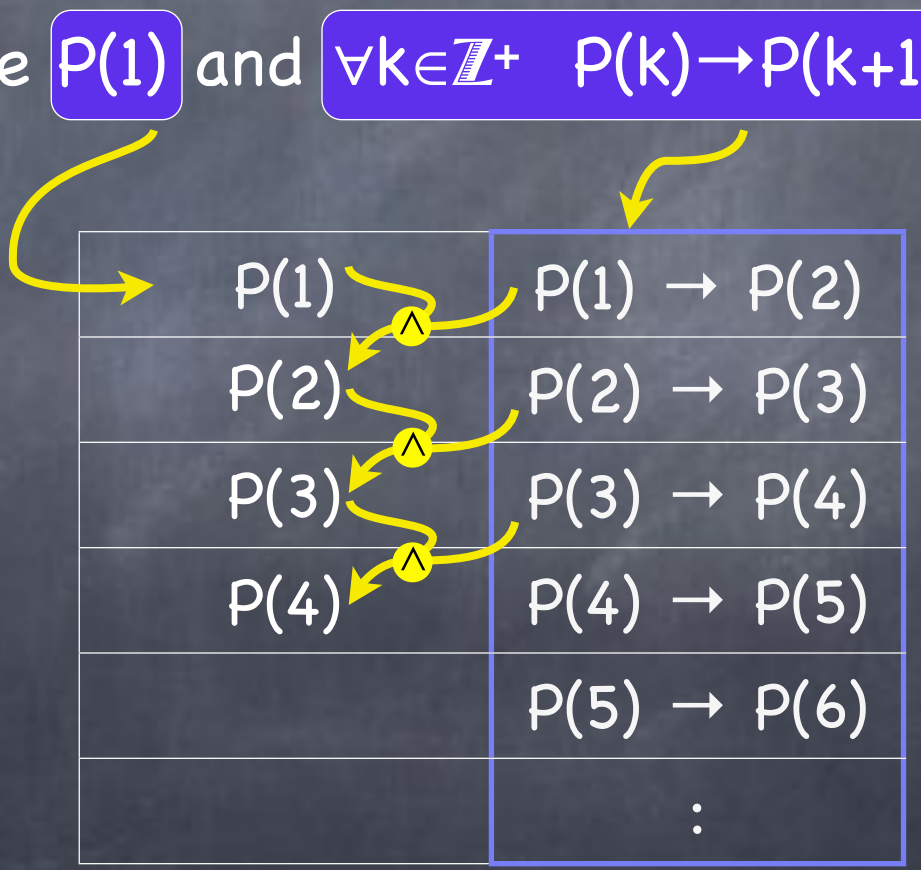

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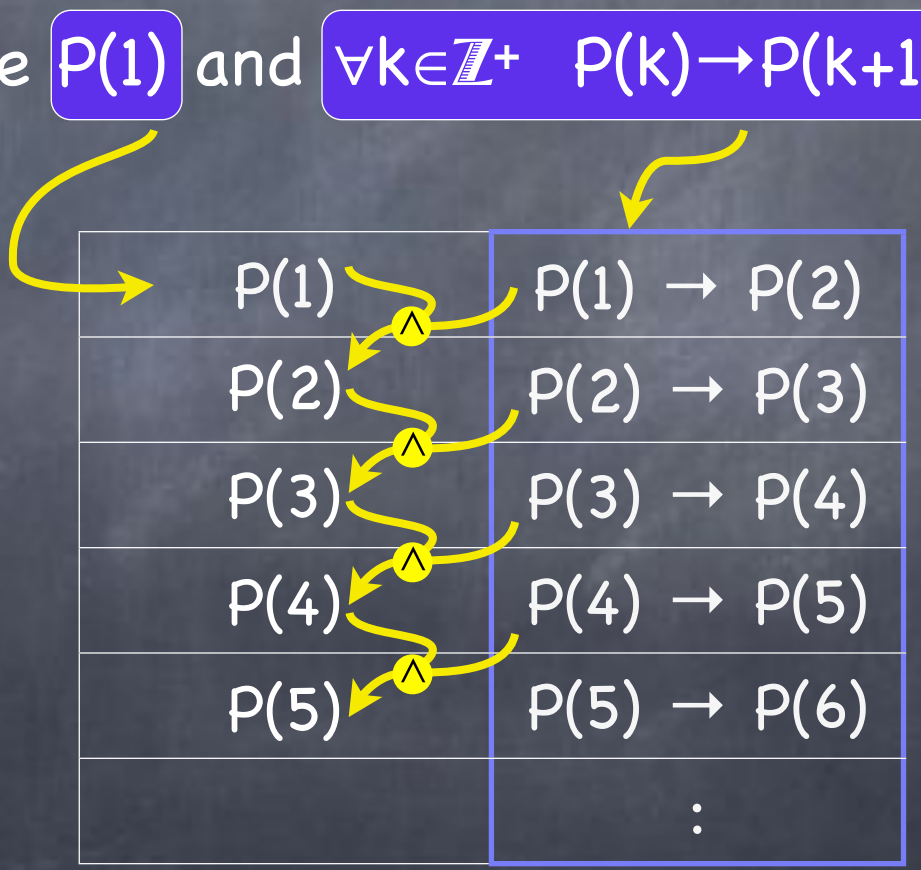

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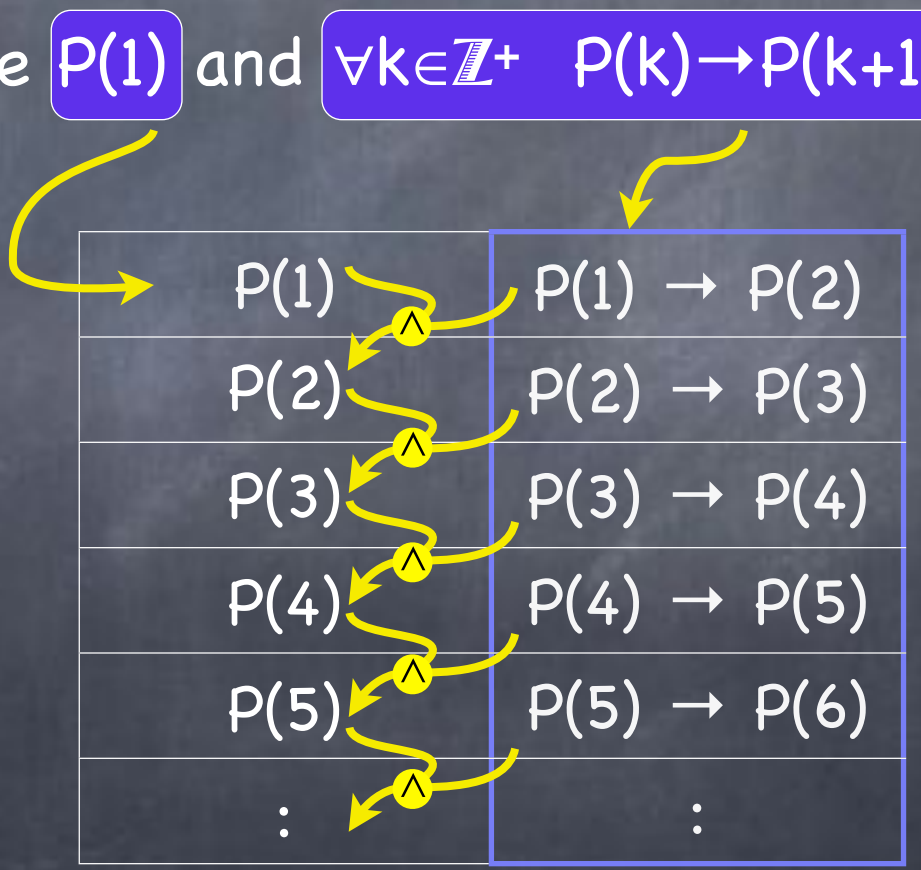

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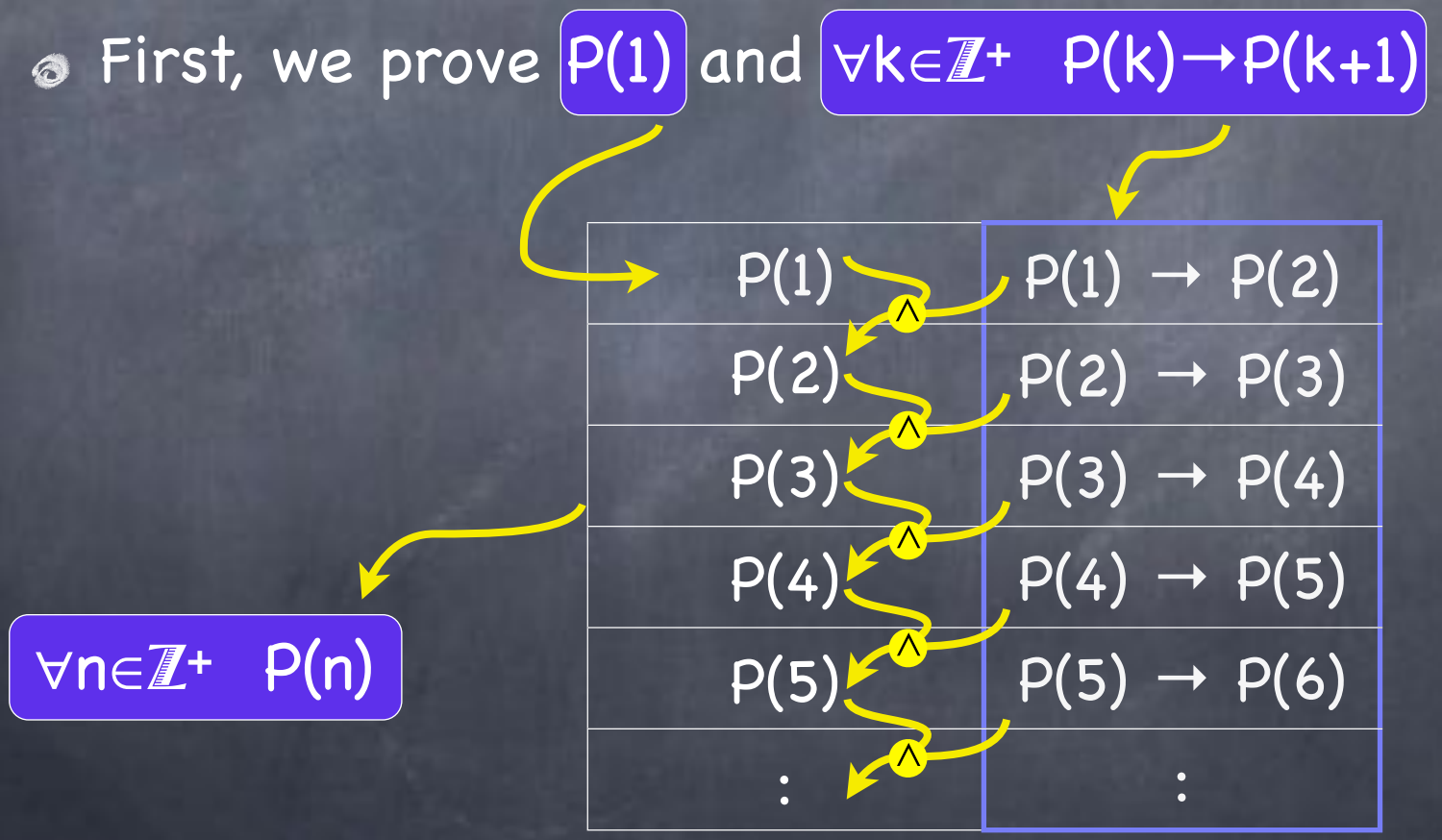

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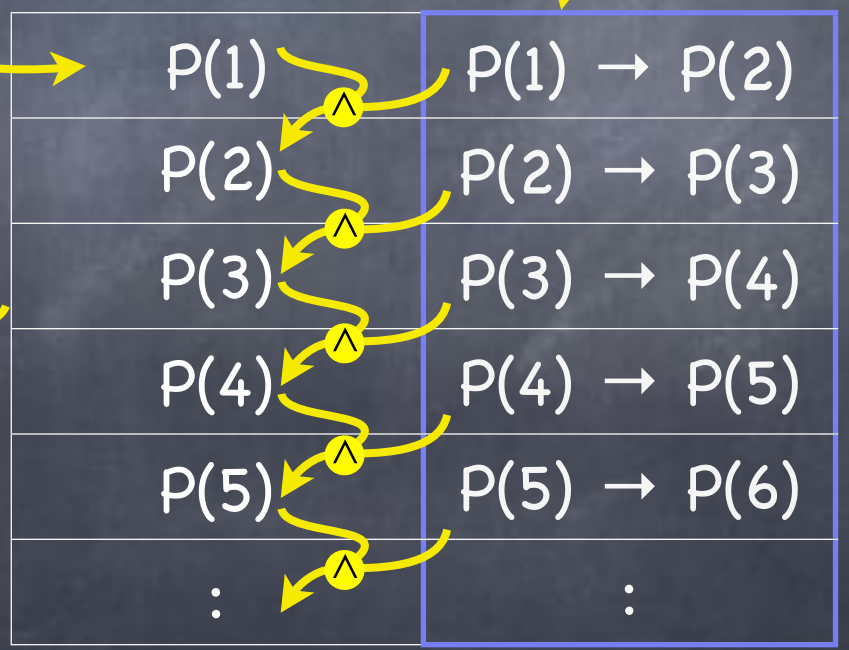
Mathematical Induction

The fact that for any n, we can run this procedure to generate a proof for P(n), and hence for any n, P(n) holds.



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- To prove $\forall n \in \mathbb{Z}^+ \;\; P(n)$:
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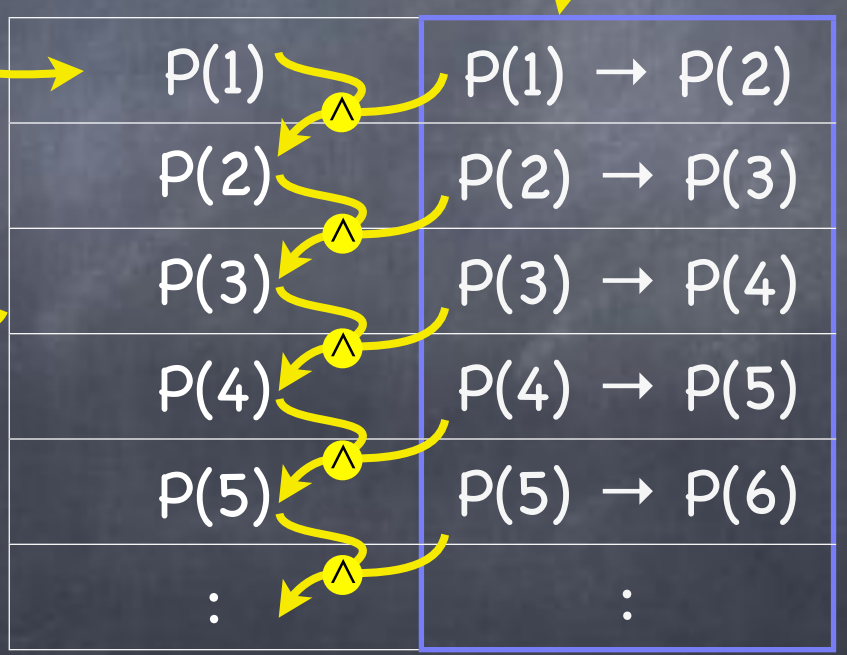
Weak Mathematical Induction

The fact that for any n, we can run this procedure to generate a proof for P(n), and hence for any n, P(n) holds.



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- To prove $\forall n \in \mathbb{Z}^+ \;\; P(n)$:
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- Some other possibilities:

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- Some other possibilities:
  - Suppose the ATM takes an $n bill and gives an $(n+2) bill

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    - To get every possible bill, start with $1 and $2 bills
  - Suppose the ATM doesn't take bills under $5
    - Print $1,…,$5 on your own. Use the ATM for the rest

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    - $2^{2(k+1)}-1 = 4.2^{2k}-1 = 4(2^{2k}-1) + 3$
      By induction hypothesis, $3 \mid 4(2^{2k}-1)$.
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      Hence, $3 \mid (2^{2(k+1)}-1)$.
  - Hence (by weak induction), $\forall n \in \mathbb{Z}^+ \;\; 3 \mid (2^{2n}-1)$