Relations

Lecture 7

Question

Question

- Pre-Lecture preparation:

  A. Read book, took quiz, and it helped
  B. Read book, took quiz, but didn't quite follow
  C. Read book, took quiz, and found it very easy
  D. Just took quiz, and I think it helped
  E. Just took quiz, but it was too easy

Question

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- Homework:

A. HW1 was easy but HW2 was hard/confusing
B. Both HW1 & HW2 were OK/easy
C. Both HW1 & HW2 were hard/confusing
D. Did not submit HW1 or HW2

Relations: Basics

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  - e.g. Likes(x,y)

x,y	Likes(x,y)
Alice, Alice	TRUE
Alice, Jabberwock	FALSE
Alice, Flamingo	TRUE
Jabberwock, Alice	FALSE
Jabberwock, Jabberwock	TRUE
Jabberwock, Flamingo	FALSE
Flamingo, Alice	FALSE
Flamingo, Jabberwock	FALSE
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  - Or a predicate over the domain S×S
  - e.g. Likes(x,y)
  - Likes = { (Alice,Alice),
    (Alice, Flamingo),
    (J'wock,J'wock),
    (Flamingo,Flamingo) }

x,y	Likes(x,y)
Alice, Alice	TRUE
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(Alice, Flamingo),
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  - More common notation:
x Likes y
    - or, $x \sqsubset y$, $x \leq y$, $x \sim y$, xLy, ...

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What is a Relation?

Many ways to look at it!

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$R \subseteq S \times S$ a set of ordered-pairs $\{ (a,b) \mid a \sqsubset b \}$	Boolean matrix, $M_{a,b} = 1$ iff $a \sqsubset b$	(directed) graph

What is a Relation?

Many ways to look at it!

$R \subseteq S \times S$
a set of
ordered-pairs
$\{ (a,b) \mid a \sqsubset b \}$

{ (A,A), (A,F),
(J,J), (F,F) }

Boolean matrix,
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(directed) graph

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	A	J	F
A	1	0	1
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(directed) graph

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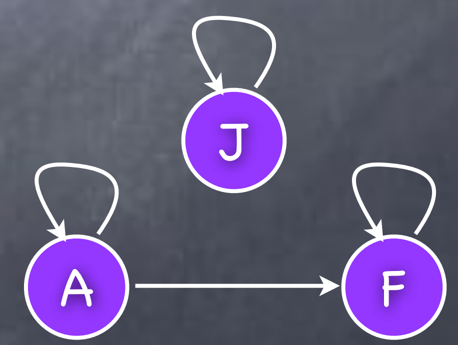

(Ir)Reflexive Relations

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- Reflexive (e.g. Knows, $\leq$)
  - The kind of relationship that everyone has with themselves

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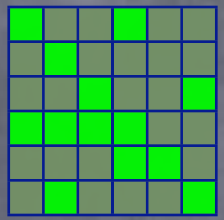

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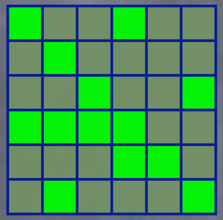

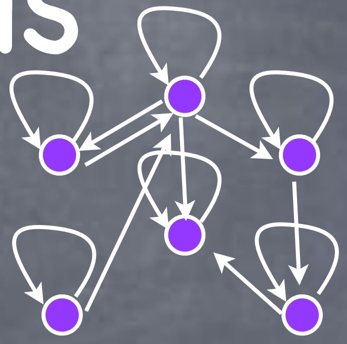

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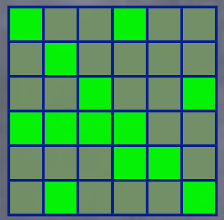

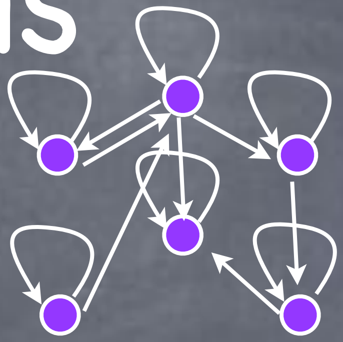

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All of diagonal included

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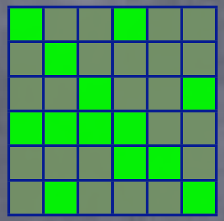

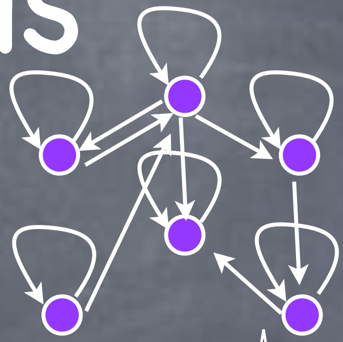

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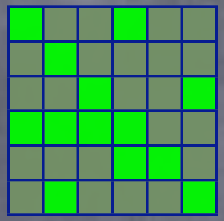

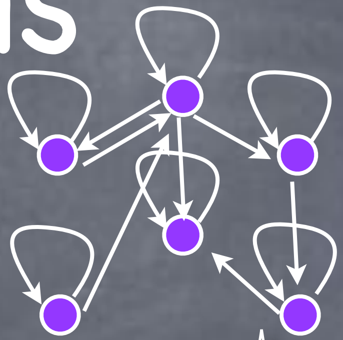

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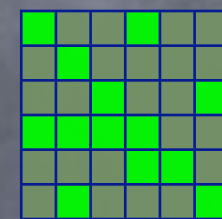

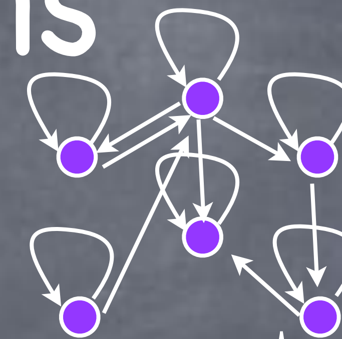

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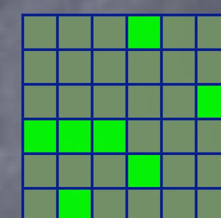

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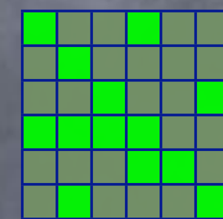
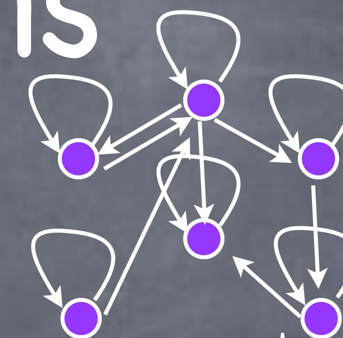


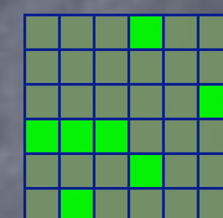


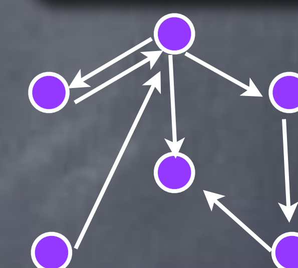
All of diagonal included

All self-loops

  - The kind of relationship that everyone has with themselves

- Irreflexive (e.g. Gave birth to, ≠)





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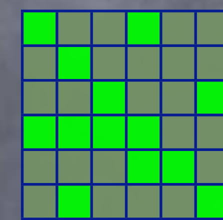

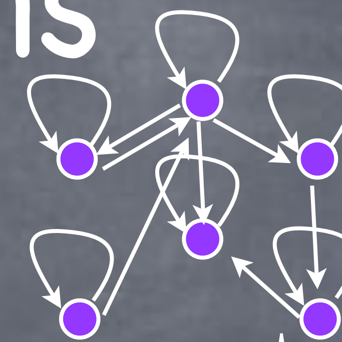

All of diagonal included

All self-loops

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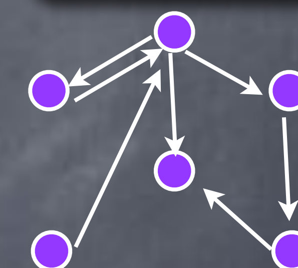
None of it



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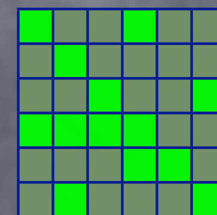

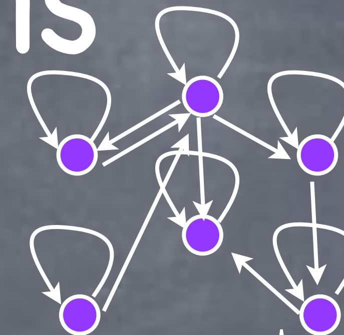

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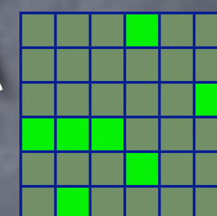

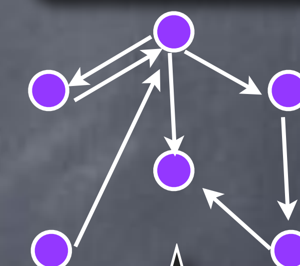

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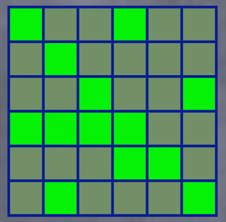

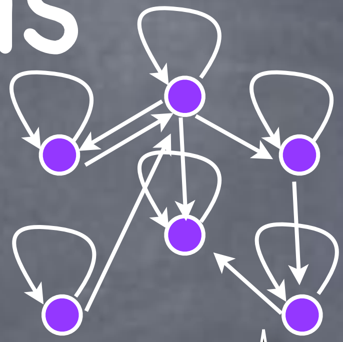

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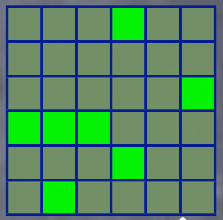

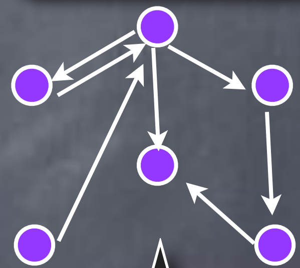

- Neither (e.g. Votes for, is a prime factor of)
  - Some, but not all, have this relationship with themselves

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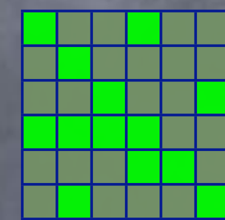

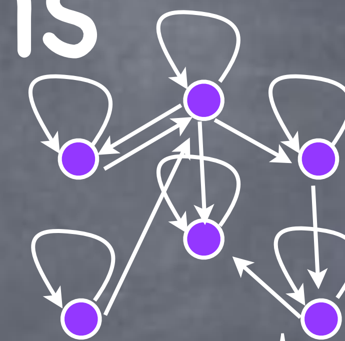

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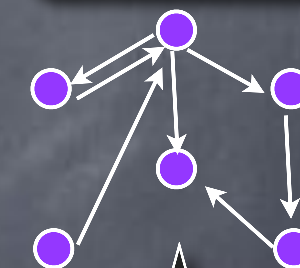

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(Anti)Symmetric Relations

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- Symmetric (e.g. Married to)
  - The relationship is reciprocated

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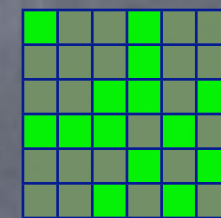

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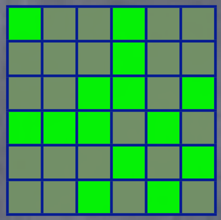

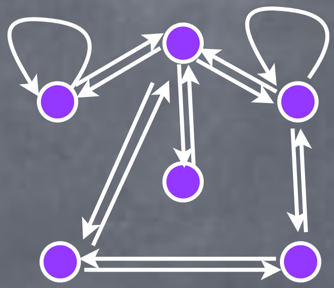

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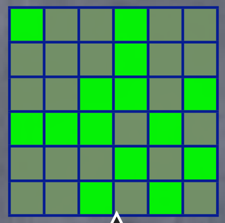

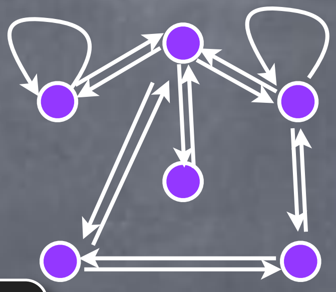

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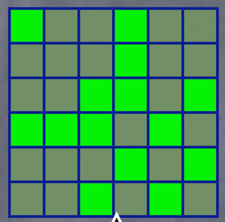

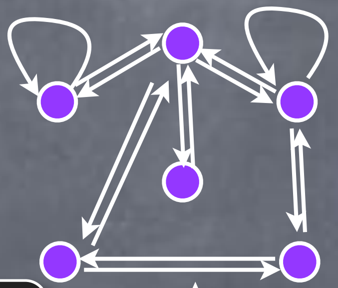

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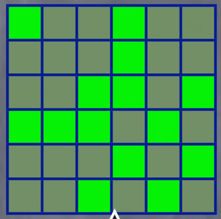

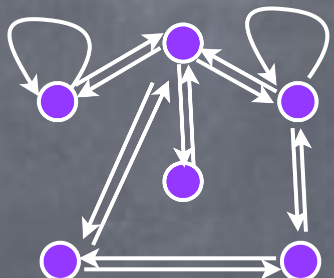

- Anti-symmetric (e.g. Parent of, divides (in $\mathbb{Z}^+$), <)
    - No reciprocation (except possibly with self)

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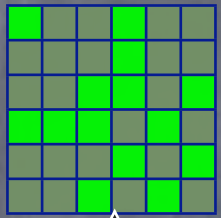

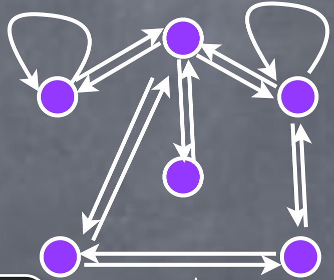

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no bidirectional edges

(Anti)Symmetric Relations

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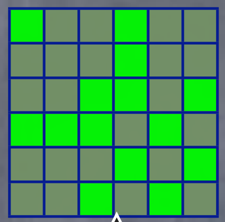
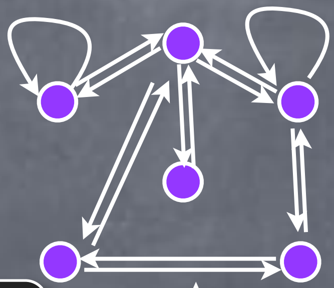
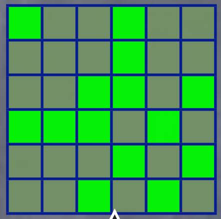
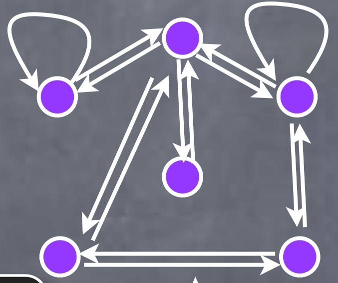




- Anti-symmetric (e.g. Parent of, divides (in $\mathbb{Z}^+$), $<$)
  - No reciprocation (except possibly with self)
- Neither (e.g. in the "circle" of)
  - Reciprocated in some pairs (with distinct members) and only one-way in other pairs

(Anti)Symmetric Relations





- Symmetric (e.g. Married to)
  - The relationship is reciprocated
- Anti-symmetric (e.g. Parent of, divides (in $\mathbb{Z}^+$), $<$)
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no bidirectional edges

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some bidirectional, some unidirectional

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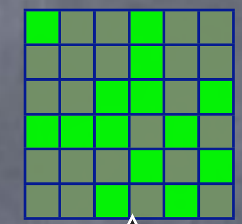
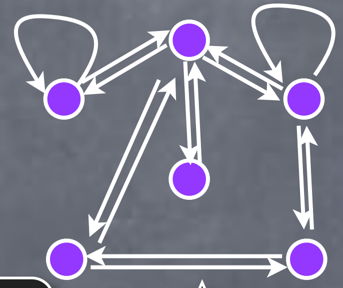


symmetric matrix



self-loops & bidirectional edges only

- Anti-symmetric (e.g. Parent of, divides (in $\mathbb{Z}^+$), <)
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no bidirectional edges

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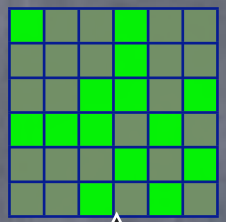
  - Reciprocated in some pairs (with distinct members) and only one-way in other pairs

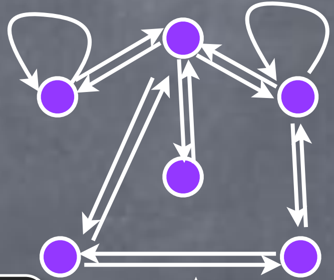
some bidirectional, some unidirectional

- Both (e.g., =)
  - Each one related only to self (if at all)

(Anti)Symmetric Relations



symmetric matrix



self-loops & bidirectional edges only

- Symmetric (e.g. Married to)
  - The relationship is reciprocated
- Anti-symmetric (e.g. Parent of, divides (in $\mathbb{Z}^+$), <)
  - No reciprocation (except possibly with self)

no bidirectional edges

- Neither (e.g. in the "circle" of)
  - Reciprocated in some pairs (with distinct members) and only one-way in other pairs

some bidirectional, some unidirectional

- Both (e.g., =)
  - Each one related only to self (if at all)

no edges except self-loops

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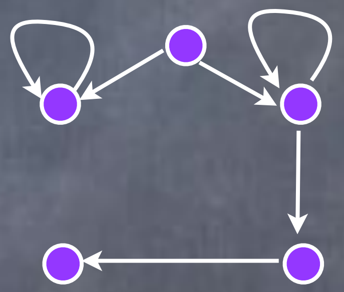

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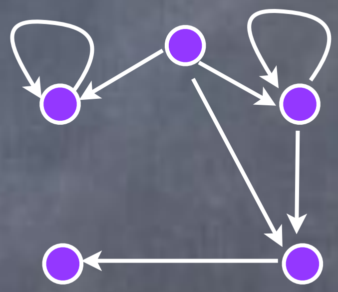

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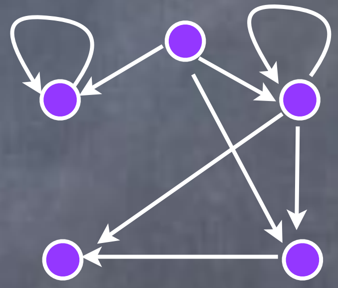

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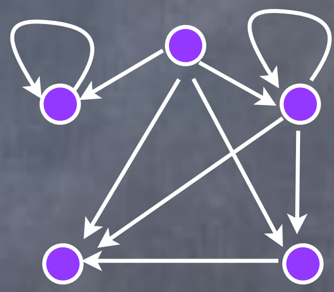

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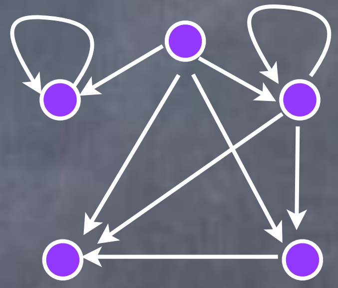

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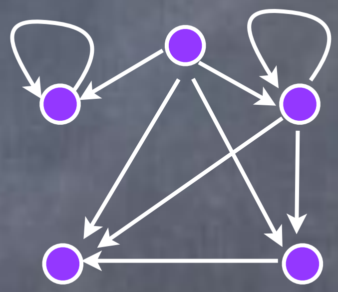

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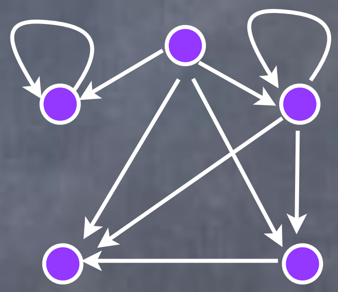

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  - "Transitive closure" of the relation is same as itself
- Intransitive: Not transitive

Question

- Suppose $\sqsubset$ is a transitive and anti-symmetric relation. Which choice is <u>impossible</u>?

  A. $a \sqsubset b$, $b \sqsubset c$, and $a \sqsubset c$
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Having a cycle: Some node s.t. you can leave it through an edge (not self-loop), move through some edges, and return to the node

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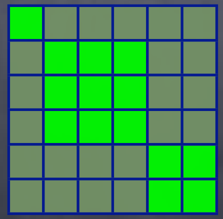

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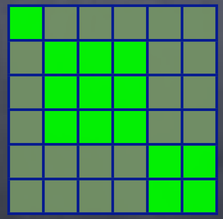

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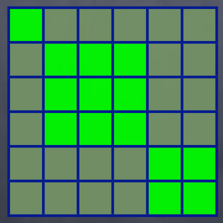

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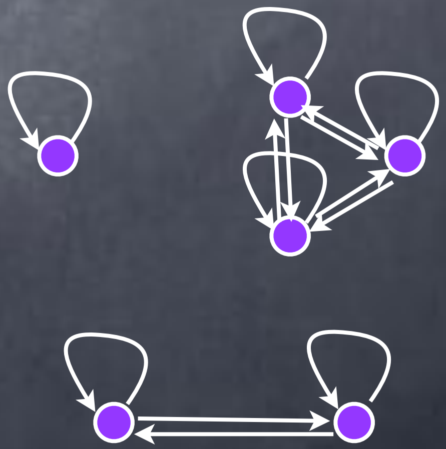

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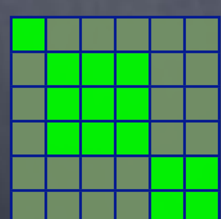

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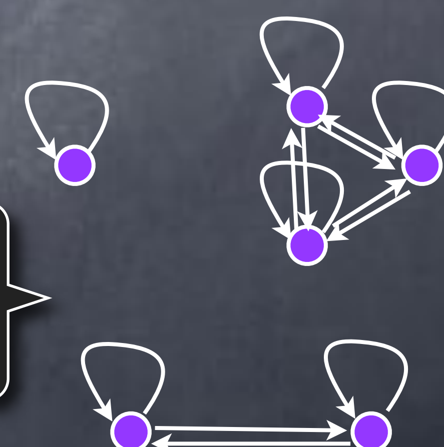
“Cliques” for each class



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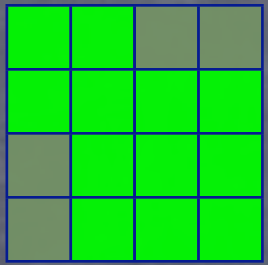
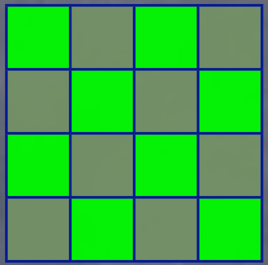
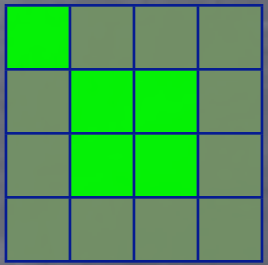


R_1



R_2



R_3

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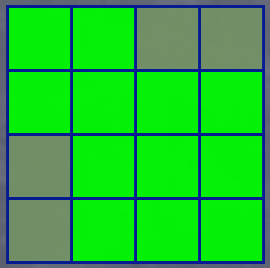

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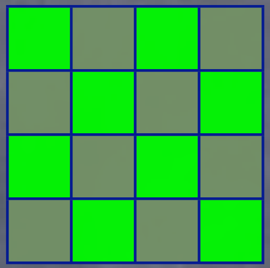
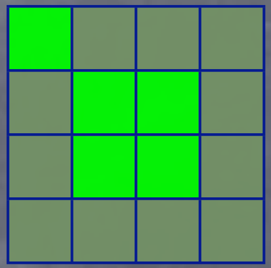


R_2



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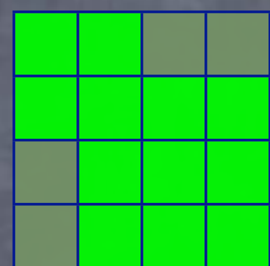
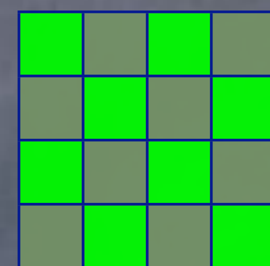
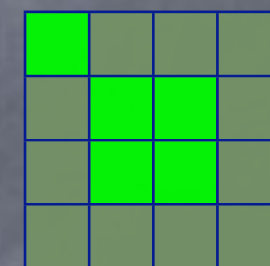


R_1

not transitive



R_2



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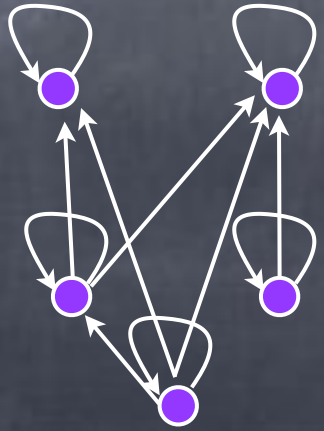

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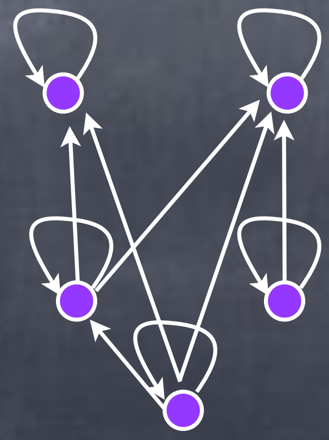

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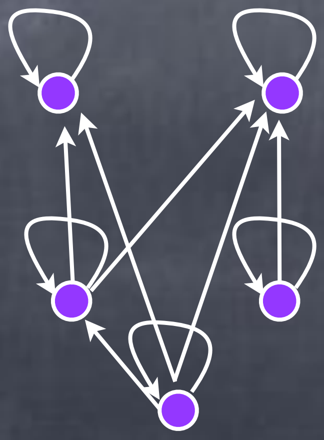

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Strict partial order: irreflexive, rather than reflexive

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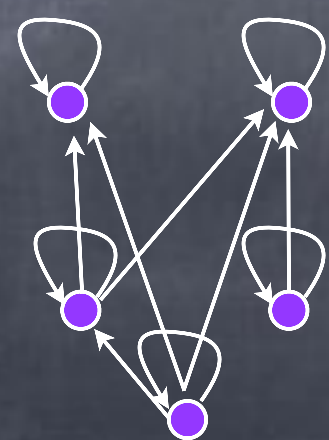

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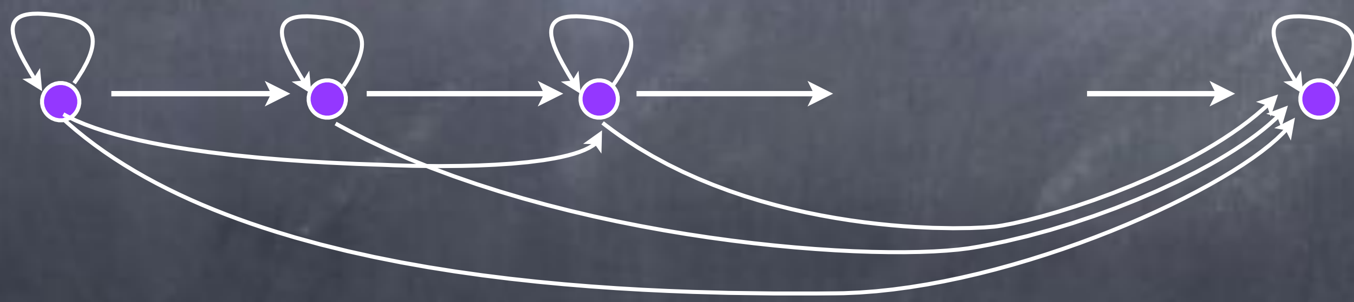

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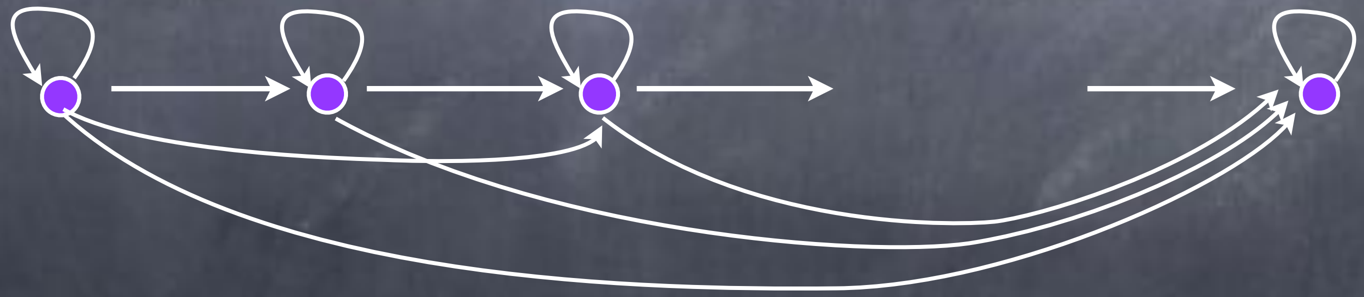

- Non-example: the relation $\subseteq$ for subsets of integers

Question

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  A. $(\forall a,b \;\; a \sqsubset b) \rightarrow \sqsubset$ is an equivalence relation
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Strict partial order $\rightarrow$ irreflexive $\rightarrow$ $\neg$ reflexive $\rightarrow$ $\neg$ partial order