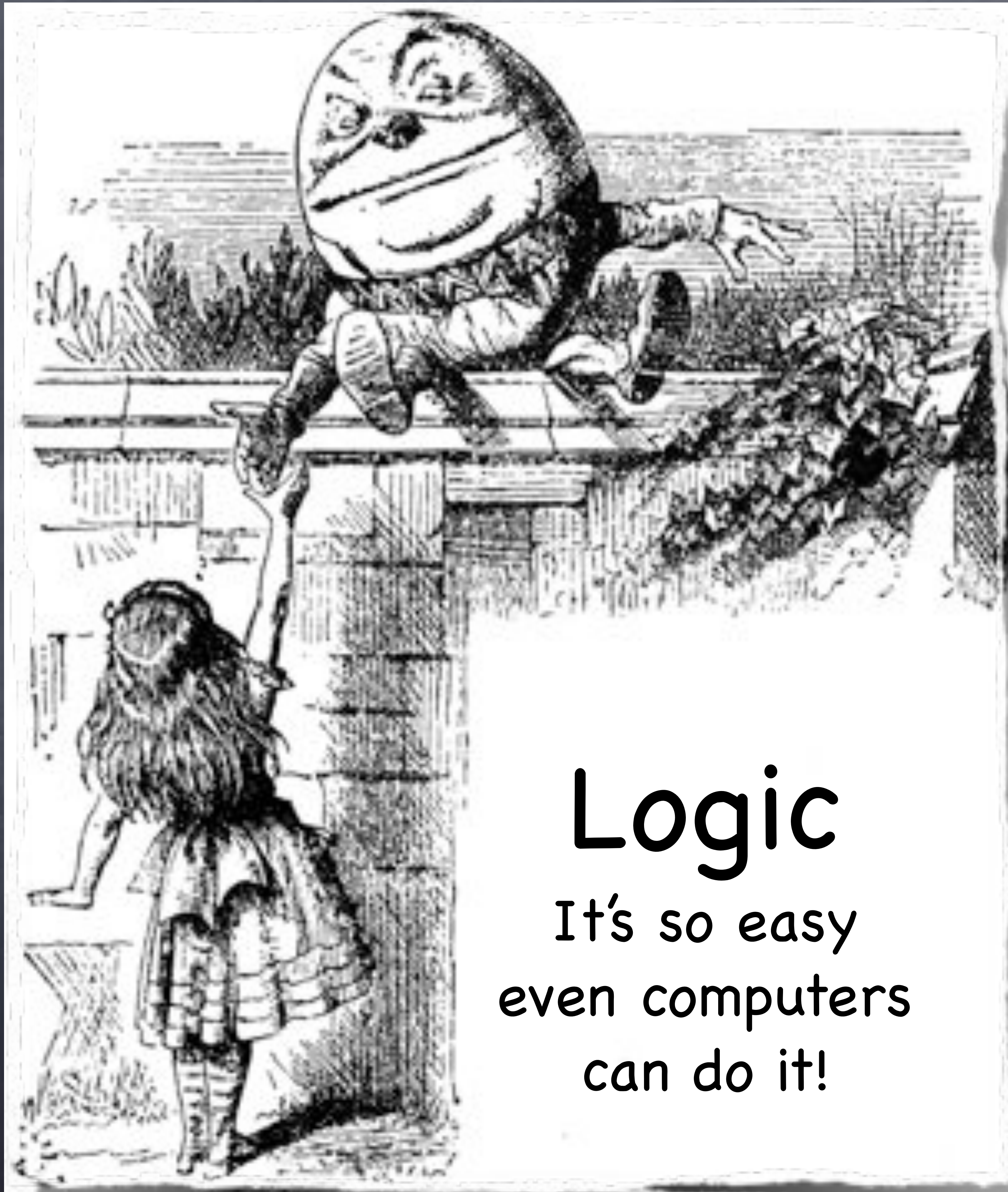

Logic

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even computers
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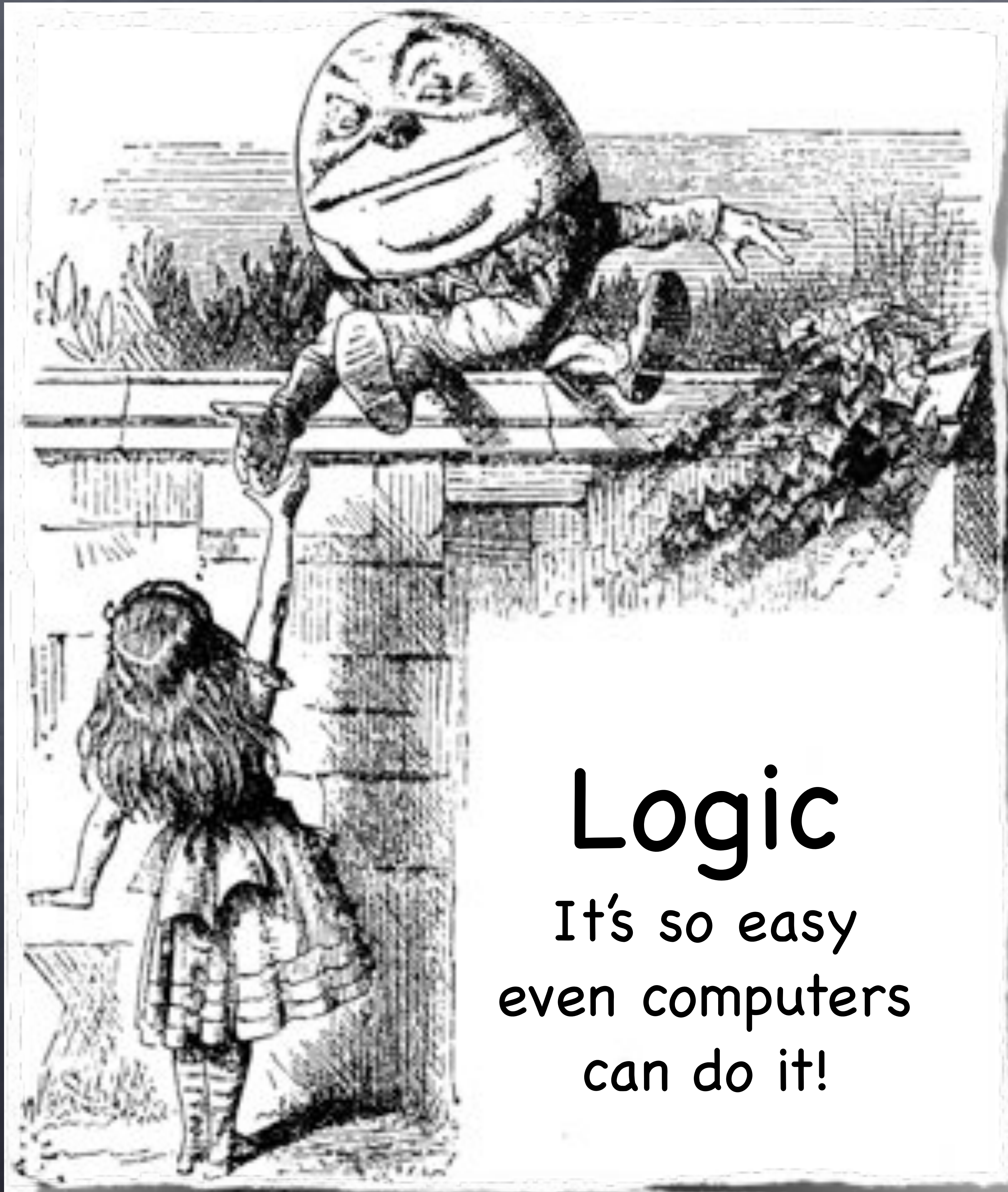

Logic

It’s so easy
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OK, It’s not
that easy!

New Kinds of Propositions

(First-Order) Predicate Calculus

x	Winged(x)	Flies(x)	Pink(x)
Alice	FALSE	FALSE	FALSE
Jabberwock	TRUE	TRUE	FALSE
Flamingo	TRUE	TRUE	TRUE

- Universal quantifier: $\forall x$ Winged(x)
  means Winged(Alice) $\wedge$ Winged(J'wock) $\wedge$ Winged(Flamingo)
- Existential quantifier: $\exists x$ Winged(x)
  means Winged(Alice) $\vee$ Winged(J'wock) $\vee$ Winged(Flamingo)

The Looking Glass

- Reflection changes T & F to F & T (resp.)
  - $\vee$ & $\wedge$ are reflected as $\wedge$ & $\vee$ (resp.)
  - $\forall$ & $\exists$ are reflected as $\exists$ & $\forall$ (resp.)

p, q  $p \wedge q$	$\neg p$, $\neg q$  $\neg p \vee \neg q$
p, q  $p \vee q$	$\neg p$, $\neg q$  $\neg p \wedge \neg q$
$\forall x\ \mathrm{Pred}(x)$	$\exists x\ \neg\mathrm{Pred}(x)$
$\exists x\ \mathrm{Pred}(x)$	$\forall x\ \neg\mathrm{Pred}(x)$

Another example

x	y	Likes(x,y)	x=y
Alice	Alice	TRUE	TRUE
	Jabberwock	FALSE	FALSE
	Flamingo	TRUE	FALSE
Jabberwock	Alice	FALSE	FALSE
	Jabberwock	TRUE	TRUE
	Flamingo	FALSE	FALSE
Flamingo	Alice	FALSE	FALSE
	Jabberwock	FALSE	FALSE
	Flamingo	TRUE	TRUE

- $\forall x,y \ (x=y) \rightarrow Likes(x,y)$
- $\forall x,y \ Likes(x,y) \rightarrow (x=y)$

Another example

x	y	Likes(x,y)	x=y
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- $\forall x,y \ \ (x=y) \rightarrow Likes(x,y)$
    - Everyone likes themselves (True)
- $\forall x,y \ \ Likes(x,y) \rightarrow (x=y)$

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- $\forall x,y \;\; (x=y) \rightarrow Likes(x,y)$
  - Everyone likes themselves (True)
- $\forall x,y \;\; Likes(x,y) \rightarrow (x=y)$
  - Everyone likes only themselves, if at all anyone (False)

Question

- Everyone who flies is winged

A. $\forall x \quad Flies(x) \lor Winged(x)$

B. $\forall x \quad Flies(x) \land Winged(x)$

C. $\forall x \quad Flies(x) \rightarrow Winged(x)$

D. $\forall x \quad Flies(x) \leftrightarrow Winged(x)$

E. $\forall x \quad Flies(x) \leftarrow Winged(x)$

Question

- Everyone who flies is winged

  A. $\forall x \;\; Flies(x) \vee Winged(x)$

  B. $\forall x \;\; Flies(x) \wedge Winged(x)$

  C. $\forall x \;\; Flies(x) \rightarrow Winged(x)$

  D. $\forall x \;\; Flies(x) \leftrightarrow Winged(x)$

  E. $\forall x \;\; Flies(x) \leftarrow Winged(x)$

- If anyone flies then he/she/it is winged

Question

- $\forall x\ \underline{P(x) \rightarrow R}$ (R independent of x)

  If for any x, P(x) holds, then R holds

  A. $(\ \forall x\ P(x)\ ) \rightarrow R$

  B. $\exists x\ P(x) \rightarrow R$

  C. $(\ \exists x\ P(x)\ ) \rightarrow R$

  D. $(\ \exists x\ P(x)\ ) \vee R$

  E. $(\ \forall x\ P(x)\ ) \wedge R$

Moving the Quantifiers

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  If for any x P(x) holds, then R holds
  - $\forall x\ \neg P(x) \vee R \equiv (\forall x\ \neg P(x)) \vee R$
    $\equiv \neg(\exists x\ P(x)) \vee R$
    $\equiv (\exists x\ P(x)) \rightarrow R$

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Moving the Quantifiers

- $\forall x\ \forall y\ P(x,y) \equiv \forall y\ \forall x\ P(x,y)$
- $\exists x\ \exists y\ P(x,y) \equiv \exists y\ \exists x\ P(x,y)$
- When R is independent of x

$\forall x\ P(x) \vee R \equiv (\forall x\ P(x)\ ) \vee R$

$\exists x\ P(x) \vee R \equiv (\exists x\ P(x)\ ) \vee R$

$\forall x\ R \rightarrow P(x) \equiv R \rightarrow (\forall x\ P(x))$

$\forall x\ P(x) \rightarrow R \equiv (\exists x\ P(x)) \rightarrow R$

$\exists x\ P(x) \wedge R \equiv (\exists x\ P(x)) \wedge R$

$\forall x\ P(x) \wedge R \equiv (\forall x\ P(x)) \wedge R$

$\exists x\ R \rightarrow P(x) \equiv R \rightarrow (\exists x\ P(x))$

$\exists x\ P(x) \rightarrow R \equiv (\forall x\ P(x)) \rightarrow R$

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- When R is independent of x

$\forall x\ P(x) \vee R \equiv (\forall x\ P(x)) \vee R$	$\exists x\ P(x) \wedge R \equiv (\exists x\ P(x)) \wedge R$
$\exists x\ P(x) \vee R \equiv (\exists x\ P(x)) \vee R$	$\forall x\ P(x) \wedge R \equiv (\forall x\ P(x)) \wedge R$
$\forall x\ R \rightarrow P(x) \equiv R \rightarrow (\forall x\ P(x))$	$\exists x\ R \rightarrow P(x) \equiv R \rightarrow (\exists x\ P(x))$
$\forall x\ P(x) \rightarrow R \equiv (\exists x\ P(x)) \rightarrow R$	$\exists x\ P(x) \rightarrow R \equiv (\forall x\ P(x)) \rightarrow R$

not the mirror!

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- $\forall x\ \forall y\ P(x,y) \equiv \forall y\ \forall x\ P(x,y)$
- $\exists x\ \exists y\ P(x,y) \equiv \exists y\ \exists x\ P(x,y)$
- When R is independent of x

$\forall x\ P(x) \lor R \equiv (\forall x\ P(x)) \lor R$

$\exists x\ P(x) \lor R \equiv (\exists x\ P(x)) \lor R$

$\forall x\ R \to P(x) \equiv R \to (\forall x\ P(x))$

$\forall x\ P(x) \to R \equiv (\exists x\ P(x)) \to R$

$\exists x\ P(x) \land R \equiv (\exists x\ P(x)) \land R$

$\forall x\ P(x) \land R \equiv (\forall x\ P(x)) \land R$

$\exists x\ R \to P(x) \equiv R \to (\exists x\ P(x))$

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- $(\forall x\ P(x)) \land (\forall x\ Q(x)) \equiv \forall x\ (P(x) \land Q(x))$

Moving the Quantifiers

- $\forall x\ \forall y\ P(x,y) \equiv \forall y\ \forall x\ P(x,y)$
- $\exists x\ \exists y\ P(x,y) \equiv \exists y\ \exists x\ P(x,y)$
- When R is independent of x

$\forall x\ P(x) \vee R \equiv (\forall x\ P(x)\ ) \vee R$

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$\forall x\ R \rightarrow P(x) \equiv R \rightarrow (\forall x\ P(x))$

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$\exists x\ P(x) \wedge R \equiv (\exists x\ P(x)) \wedge R$

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- $(\forall x\ P(x)) \wedge (\forall x\ Q(x)) \equiv \forall x\ (P(x) \wedge Q(x))$
  - But $(\forall x\ P(x)) \vee (\forall x\ Q(x)) \not\equiv \forall x\ (P(x) \vee Q(x))$

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- $\forall x\ \forall y\ P(x,y) \equiv \forall y\ \forall x\ P(x,y)$
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$$\forall x\ P(x) \lor R \equiv (\forall x\ P(x)\ ) \lor R$$
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$$\exists x\ P(x) \land R \equiv (\exists x\ P(x)) \land R$$
$$\forall x\ P(x) \land R \equiv (\forall x\ P(x)) \land R$$
$$\exists x\ R \rightarrow P(x) \equiv R \rightarrow (\exists x\ P(x))$$
$$\exists x\ P(x) \rightarrow R \equiv (\forall x\ P(x)) \rightarrow R$$

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- $(\forall x\ P(x)) \land (\forall x\ Q(x)) \equiv \forall x\ (P(x) \land Q(x))$
  - But $(\forall x\ P(x)) \lor (\forall x\ Q(x)) \not\equiv \forall x\ (P(x) \lor Q(x))$
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Moving the Quantifiers

- $\forall x\ \forall y\ P(x,y) \equiv \forall y\ \forall x\ P(x,y)$
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Question

- $\forall x,y \ \ Pink(x) \rightarrow Flies(y)$

A. If anyone is pink, he/she/it flies

B. If no one flies, someone is not pink

C. If anyone is pink, everyone flies

D. If anyone is pink, someone flies

E. If everyone is pink, everyone flies

Answer

Answer

- $\forall x \ \ \underline{\forall y \ \ P(x) \rightarrow Q(y)}$
  - $\forall x \ \ \underline{P(x) \rightarrow (\forall y \ Q(y))}$

Answer

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Answer

- $\forall x \quad \underline{\forall y \quad P(x) \rightarrow Q(y)}$
  - $\forall x \quad \underline{P(x) \rightarrow (\forall y \; Q(y))}$
- $\forall x \quad P(x) \rightarrow R \equiv (\exists x \; P(x)) \rightarrow R$
- $\underline{\forall x,y \quad P(x) \rightarrow Q(y)} \equiv \underline{(\exists x \; P(x)) \rightarrow (\forall y \; Q(y))}$

Example

Example

- "If someone flies, everyone flies" is the same as which one(s)?

1. $(\exists x \; Flies(x)) \rightarrow (\forall x \; Flies(x))$
2. $\forall x,y \; Flies(x) \leftrightarrow Flies(y)$
3. $\exists x \; \forall y \; Flies(x) \leftrightarrow Flies(y)$
4. $\exists x \; \forall y \; Flies(x) \rightarrow Flies(y)$

Example

- “If someone flies, everyone flies” is the same as which one(s)?

  1. $(\exists x \text{ Flies}(x)) \rightarrow (\forall x \text{ Flies}(x))$
  2. $\forall x,y \text{ Flies}(x) \leftrightarrow \text{Flies}(y)$
  3. $\exists x\ \forall y \text{ Flies}(x) \leftrightarrow \text{Flies}(y)$
  4. $\exists x\ \forall y \text{ Flies}(x) \rightarrow \text{Flies}(y)$

Always true!
(a.k.a, tautology)
But why?

Proofs: Logic in Action

Using Logic

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- Logic is used to deduce results in any (mathematically defined) system
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  - But can be automated if the system is relatively simple
  - Artificial Intelligence!

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  - For us, numbers (reals, integers, rationals) and other systems like sets, graphs, functions, ...
- Goal: Use logical operations to establish the truth of a given proposition, starting from the axioms (or already proven propositions) in a system

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- Definition: An integer x is said to be odd if there is an integer y s.t. $x=2y+1$
  - $\mathrm{Odd}(x) \equiv \exists y \in \mathbb{Z}\ (x = 2y+1)$

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- Proof: (should be written in more readable English)

Example

- Def: $Odd(x) \equiv \exists y \in \mathbb{Z}\ (x = 2y+1)$
- Proposition: $\forall x \in \mathbb{Z}\ Odd(x) \rightarrow Odd(x^2)$
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  - Then by definition, $\exists y \in \mathbb{Z}$ $x=2y+1$.

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  - Let x be an arbitrary element of $\mathbb{Z}$
  - Suppose $Odd(x)$. Then, we need to show $Odd(x^2)$.
  - Then by definition, $\exists y \in \mathbb{Z}\ x=2y+1$.
  - Then, $x^2 = (2y+1)^2$
    $= 4y^2 + 4y + 1$
    $= 2(2y^2+2y) + 1.$ From arithmetic.

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    $= 2(2y^2+2y) + 1.$ From arithmetic.
  - $\exists w \in \mathbb{Z}\ (2y^2+2y)=w.$ From arithmetic.
  - Hence, $\exists w \in \mathbb{Z}\ x^2 = 2w+1$
  - By definition $\mathrm{Odd}(x^2)$. QED.

Template

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- To prove $\forall x\ P(x) \rightarrow Q(x)$

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  - Assume $P(x)$ holds (call this p_0)
  - Derive propositions $p_1, p_2, \ldots, p_n$ where for each i, either p_i is an axiom or an already proven proposition in the system, or $(p_0 \wedge p_1 \wedge \ldots \wedge p_{i-1}) \rightarrow p_i$

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    - Usually one or two propositions so far imply the next
  - p_n should be Q(x).
- More generally, to prove $\forall x\, P(x)$, will typically need to use system-specific arguments

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- e.g. to prove $\neg \forall x\, P(x)$

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  - (May not be able to find one, but still show one exists!)
- e.g. to prove $\neg\, \forall x\, P(x)$
  - prove $\exists x\, \neg P(x)$: enough to demonstrate an x s.t. $P(x)$ is false

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      - e.g. $\forall x,y \in \mathbb{Z}^+\ \ x.y > 25 \rightarrow (x \geq 6) \vee (y \geq 6)$

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        - $\forall x,y \in \mathbb{Z}^+\ \ (x<6) \land (y<6) \rightarrow x.y \leq 25$

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    - $\exists x\ P(x) \rightarrow Q(x) \equiv \exists x\ \neg Q(x) \rightarrow \neg P(x)$

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  - Let x be an arbitrary element
    Show $Q(x) \rightarrow P(x)$
    Show $Q(x)$ holds
    Then $P(x)$ Because, $(Q(x) \wedge (Q(x) \rightarrow P(x))) \rightarrow P(x)$.

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At this point, we have **reduced** the problem of proving P(x) to the problem of proving Q(x)

- $\exists x\ \neg Q(x)$
  - Show $\exists x\ P(x) \lor \neg Q(x)$
    Show $\forall x\ \neg P(x)$

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At this point, we have **<u>reduced</u>** the problem of proving P(x) to the problem of proving Q(x)

- $\exists x\ \neg Q(x)$
  - Show $\exists x\ P(x) \lor \neg Q(x)$
    Show $\forall x\ \neg P(x)$

If we demonstrate an element x s.t. $P(x) \lor \neg Q(x)$ holds, now enough to show that for <u>that x</u>, P(x) holds

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    Show $Q(x) \rightarrow P(x)$
    Show $Q(x)$ holds
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At this point, we have **reduced** the problem of proving P(x) to the problem of proving Q(x)

- $\exists x\ \neg Q(x)$
  - Show $\exists x\ P(x) \vee \neg Q(x)$
    Show $\forall x\ \neg P(x)$
  - Show $\forall x\ \neg Q(x)$ (Much more than needed, but OK)

If we demonstrate an element x s.t. $P(x) \vee \neg Q(x)$ holds, now enough to show that for that x, P(x) holds

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- $\neg\ \forall x\ P(x) \equiv \exists x\ \neg P(x)$

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