Homework 7

Discrete Structures
CS 173 [B] : Fall 2012

Released: Fri Oct 26
Due: Fri Nov 2, 5:00 PM

Attention: Part of Homework 7 is on Moodle. The rest of the homework problems are given below. They need to be submitted in hardcopy.

1. Prove using induction that consecutive Fibonacci numbers are co-prime. [15 points]

2. Recursive definition and inequality [15 points]
   Let’s define a function \( G : \mathbb{N} \rightarrow \mathbb{N} \) by the following recursive definition:
   - \( G(0) = 0 \)
   - \( G(n) = G(\lfloor \frac{n}{2} \rfloor) + n \), for integers \( n > 0 \)

   Use strong induction to prove that \( G(n) < 2n \) for every integer \( n \geq 1 \).

3. Fibonacci Trees. [10 points]
   Consider the following recurrence: A fibonacci tree is defined recursively as follows
   - The fibonacci trees \( FT_0 \) and \( FT_1 \) each consists of a single node.
   - The fibonacci tree \( FT_n \) for \( n \geq 2 \) consists of a root with two subtrees, the left subtree is \( FT_{n-1} \) and the right subtree is \( FT_{n-2} \)

   Using this definition, answer the following.
   (a) Draw \( FT_2 \), \( FT_3 \) and \( FT_4 \).
   (b) As a function of \( n \), what is the height of \( FT_n \)?
   (c) Prove the above by induction.

4. Golden Ratio [10 points]
   Define a function \( g : \mathbb{N} \rightarrow \mathbb{R} \) recursively as follows:
   - \( g(1) = 1 \)
   - \( g(n + 1) = 1 + \frac{1}{g(n)} \) for all integers \( n \geq 1 \)

   Recall that the Fibonacci numbers are defined recursively as follows:
   - \( F_0 = 0, F_1 = 1 \)
   - \( F_n = F_{n-1} + F_{n-2} \) for each integer \( n \geq 2 \)
Use induction to prove that \( g(n) = \frac{F_{n+1}}{F_n} \) for all \( n \in \mathbb{Z}^+ \).

Comment: As \( n \) tends to infinity, \( g(n) \) tends to the positive solution of the quadratic equation given by \( x = 1 + \frac{1}{x} \). This number, \( \frac{1 + \sqrt{5}}{2} \approx 1.618 \) is sometimes called the “golden ratio.”

5. Context-Free Grammar. [20 points]

Consider the following grammar \( G \) whose set of non-terminals is \( N = \{ S, A, B \} \), the set of terminals is \( \Sigma = \{ a, b \} \), starting symbol \( S_0 \) is \( S \), and the set of production rules \( P \) is given by:

- \( S \rightarrow ASA | SBS | aBaB | AbAb \)
- \( A \rightarrow a \)
- \( B \rightarrow bb \)

(a) Draw two examples of trees of height two that represent strings of terminals in the language generated by \( G \). Note that grammar trees that represent strings of terminals should not have non-terminals as their leaves.

(b) Prove that any strings of terminals generated by \( G \) will always have even numbers of both \( a \)'s and \( b \)'s.

Hint: You should prove a stronger statement: any tree of height \( h > 1 \) generated by \( G \), with any of the non-terminals as start symbol, even if it has non-terminals at the leaves, has an even number of \( a \)'s and an even number of \( b \)'s (plus possibly some non-terminals). To prove this statement, use induction on trees, with the height as the induction variable.

Extra Credit Problems

6. Recursive definition: Set of points. [Extra credit]

The sets \( T_n \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+ \) are defined as below.

- \( T_1 = \{(1,1)\} \).
- \( T_n = \{(x+1, y+1)| (x, y) \in T_{n-1}\} \cup \{(x+2, y)| (x, y) \in T_{n-1}\} \cup \{(x, y+2)| (x, y) \in T_{n-1}\} \) for all integers \( n > 1 \).

Let \( T = T_1 \cup T_2 \cup T_3 \cup \cdots \) be an infinite union of the sets \( T_n \) (i.e., \( (x, y) \in T \) iff \( \exists n \geq 1 \) such that \( (x, y) \in T_n \)).

Give a simple closed-form definition for the \( T \). Give both a precise definition using set-builder notation and also an informal geometrical description using a picture (as a set of points on the plane). You don’t have to submit a proof that your answer is correct.

7. Arrangement of Lines in a Plane [Extra credit]

Consider \( n \) lines in the plane, arranged such that no two lines are parallel and no three lines pass through a common point.

(a) Find the number of regions into which a plane is partitioned by \( n \) such lines where \( n = 0, 1, 2, 3, 4 \).

(b) By examining solutions of part (a), find a formula for the number of regions into which \( n \) lines can partition the plane.

(c) Use induction to prove the formula derived in part (b).

[Hint: If a new line is not parallel to any of the existing lines, it will intersect each of them. Can you account for the increase in the number of regions by looking at the intersections?]
8. Bit Strings without Consecutive Zeros  

A *bit-string* is simply a finite sequence of zeroes and ones. For the purposes of this problem, strings will always have length \( \geq 1 \), i.e. no zero-length strings.

Let \( A_n \) be the number of strings of length \( n \) that end in a 1, and have no two consecutive zeros. Let \( B_n \) be the number of strings of length \( n \) that end in a 0, and have no two consecutive zeros. Thus \( A_1 = 1 \) and \( B_1 = 1 \). \( A_2 = 2 \) (strings 01 and 11) and \( B_2 = 1 \) (string 10).

(a) List \( A_n \) and \( B_n \) for \( n = 3, 4, 5 \).

(b) Give recursive definitions for \( A_n \) and \( B_n \) in terms of \( A_{n-1} \) and \( B_{n-1} \) (each one possibly using both of them).

(c) What is \( A_n \) and \( B_n \) in terms of the Fibonacci numbers?

(d) How many bit-strings of length \( n \) are there in which there are no two consecutive zeros?