Homework 2

Discrete Structures
CS 173 [B] : Fall 2012

[Released: Tue Sep 11. Due: Wed Sep 19, 3:00 PM]

1. To prove that $\forall x\ P(x) \not\to Q(x)$, which approach(es) is/are correct? (5 points)
   (a) Show that $P(0) \not\to Q(0)$.
   (b) Show that $\forall x\ \neg Q(x)$.
   (c) Show that $\forall x\ P(x) \wedge \neg Q(x)$.
   (d) Show that $\forall x\ P(x)$ and show that $\exists x\ \neg Q(x)$.

2. Prove that $\exists x \forall y\ P(x) \to P(y)$ is true no matter what P is (assuming that the domain is non-empty). [Hint: consider two cases, depending on whether $\forall y\ P(y)$ is true or false.] (15 points)

3. Prove the following claims: (50 points)
   (a) For any integers k, m and n, if $k|m$ and $m|n$ then $k|n$.
   (b) A pair of real numbers (x, y) is said to be an *interval* if $x \leq y$. An interval (x, y) is said to *contain* an interval (p, q) if $x \leq p$ and $q \leq y$. Using this definition, prove or disprove the following:
      i. For any intervals (a, b), (c, d), and (e, f), if (a, b) contains (c, d) and (c, d) contains (e, f), then (a, b) contains (e, f).
      ii. For any intervals (a, b), (c, d), and (e, f), if (a, b) contains (c, d) and (a, b) contains (e, f), then either (c, d) contains (e, f) or (e, f) contains (c, d) (or both).
   (c) For any integer x, if $\mid x + 7 \mid > 8$, then $\mid x \mid > 1$.
   (d) If $a^2 > 25$ and $b^2 > 49$, then $|ab| > 35$.
   [You can use the following: $\forall x \in \mathbb{Z}\ x^2 = |x^2|$. $\forall w, x, y, z \in \mathbb{Z}\ (|w| < |x|) \wedge (|y| < |z|) \to |wy| < |xz|$. You can also use the semantics of squaring and that of the operators $>, < . =$ etc. for integers, like, for e.g., $\neg(a < b) \leftrightarrow ((a > b) \vee (a = b))$.]

4. Consider the following statement: If m and n are positive integers such that mn is even, then m is even or n is even. (20 points)
   (a) Write a proof for the above statement (in English sentences).
   (b) Translate the statement into predicate logic (in terms of the predicate `Even`). Then prove the statement in question using logical equivalences, and propositions from arithmetic (explicitly indicate when you use the latter). (Hint: Prove the contrapositive.)

5. Over $\mathbb{Z} \times \mathbb{Z}^+ \times \mathbb{Z}^+$, define the predicate $M(x, a, b)$ to be true iff $\gcd(a, b) \mid x$ (i.e., x is a multiple of $\gcd(a, b)$). Also define the predicate $L(x, a, b)$ to be true iff $\exists r, s \in \mathbb{Z}\ \ x = ra + sb$. (This says that x is in the "lattice" generarted by a and b.) Prove that

$$\forall x \in \mathbb{Z}, \forall a, b \in \mathbb{Z}^+\ M(x, a, b) \leftrightarrow L(x, a, b).$$

   [Hint: You will have to show both $L(x, a, b) \to M(x, a, b)$ and $M(x, a, b) \to L(x, a, b)$. The first one you should be able to show from the definitions. For the other direction, you can use the fact (implied by the Euclidean algorithm for GCD) that $\forall p, q \in \mathbb{Z}^+\ \exists u, v \in \mathbb{Z}\ \ \gcd(p, q) = up + vq$.] (20 points)