Homework 10

Discrete Structures
CS 173 [B] : Fall 2012

Released: Saturday Dec 1
Due: Friday Dec 7, 5:00 PM

1. **Combinations** [20 points]

   A television director is scheduling a certain sponsor's commercials for an upcoming broadcast. There are six slots available for commercials. In how many ways may the director schedule the commercials, in each of the following cases?

   (a) If the sponsor has six different commercials, each to be shown once?

   (b) If the sponsor has three different commercials, each to be shown twice?

   (c) If the sponsor has three different commercials, the first of which is to be shown three times, the second two times, and the third once?

   (d) If the sponsor has three different commercials, each of which should be shown at least once and at most three times.

2. **Cards** [20 points]

   In a deck of cards, there are 13 different *kinds* of cards (twos, threes, fours, fives, sixes, sevens, eights, nines, tens, jacks, queens, kings, and aces) and four different *suits* (spades, clubs, hearts, and diamonds). A deck has 52 cards in all: one card of each kind in each suit.

   A *hand* consists of some number of cards drawn from the deck. In this problem we consider hands of 5 cards. The order of the cards in a hand doesn't matter.

   (a) How many hands of five cards are there?

   (b) How many hands of five cards contain four cards of one kind?

   (c) How many hands of five cards contain three cards of one kind and two of another kind?

   (d) How many hands of five cards have five cards of the same suit?

3. **Lottery** [20 points]

   Counting is intimately connected to computing the *probability* of various events. In this problem we shall use counting to calculate the probability of winning certain kinds of lotteries.

   (a) There are many lotteries that award enormous prizes to people who correctly choose a set of six numbers out of the first n positive integers, where n is usually between 30 and 60.

      i. How many ways can one form a subset of size 6 from the set $[40] = \{1, \ldots, 40\}$?

      ii. If the correct set is chosen "uniformly at random" out of all possible subsets of [40] of size 6, then no matter what subset the player chooses, the *probability* that the set chosen as the correct set matches player's choice is $1/m$, where m is the number from the last part. Calculate this probability.

(b) In a different kind of lottery, players win a large prize when they pick four digits (between 0 and 9) that match, in the correct order, four digits selected by a random mechanical process. A smaller prize is won if only three digits are matched.

   i. How many ways can the correct sequence of four digits be chosen? Use this to compute the probability of a player winning the large prize no matter what the player's choice is (as in the previous problem).

   ii. For any sequence of four digits that a player picks, how many correct sequences are there which would result in the player winning a small prize? Use this to compute the probability that a player wins the small prize, as $\frac{\text{number of correct sequences giving a small prize}}{\text{number of all possible correct sequences}}$.

4. **Sets of Sets** [10 points]

   Let us define a function $g : \mathbb{N}^+ \times \mathbb{N}^+ \to \bigcup_{m \in \mathbb{N}^+} \mathbb{P}(\mathbb{Z}_m)$, where $g(m,p) = \{[a]^p \;:\; [a] \in \mathbb{Z}_m\}$. (Recall that $\mathbb{Z}_m = \{[0], \ldots, [m-1]\}$, and for $[a],[b] \in \mathbb{Z}_m$, $[a]^p = [b]$ iff $a^p \equiv b \pmod{m}$.)

   (a) What is $g(m,p)$? Describe in 2-3 sentences.

   (b) Calculate $g(17,2)$, $g(17,3)$ and $g(m,1)$.

5. **A Finite State Machine** [15 points]

   Consider a machine which has n bits of memory. Its state is determined by the contents of its memory. This machine takes an input from the set $\{1, \ldots, n\}$: on input i, it toggles its i^{th} bit. (Toggling a bit changes its value from 0 to 1 or from 1 to 0.)

   (a) Draw the state diagram for this machine for $n = 2$. (Be sure to label each state with the corresponding 2 bits, and to label each arc with the input that causes the machine to make the corresponding transition.)

   (b) What graph does the state-diagram for this machine (for a general value of n) resemble? You can treat pairs of directed edges between same two states, but pointing in opposite directions as a single undirected edge.

6. **Finite State Machines and Strings** [15 points]

   A state machine with a start state and one or more final states (the start state may or may not be a final state) is said to *accept* a string if, on being given this string as input (character by character), the machine can move from the start state to a final state so that at every step, the label on the arc traversed is the same as the input character at that step.

   Draw a state machine that accepts exactly those strings which consist only of a's and b's such that the number of a's is even and the number of b's is odd.

   [Hint: How many states must this machine have?]