CS 173 [B], Fall 2012
Midterm 2: Take-Home Exam
13-15 November 2012

NAME:

NETID (e.g. hpotter23, not 123987654) **:**

DISCUSSION SECTION (11:00, 12:00, 1:00, 2:00, 3:00 or 4:00) **:**

Problem	1	2	3	4	5	6	7	8	9	Extras	Total
Possible	10	6	6	20	9	10	12	12	15	(20)	100
Score											

This is a take-home exam.
The exam relies on a strict honor code.
Please read and sign the following honor-pledge.

Honor Pledge

I understand that this exam tests my individual preparation and effort, and that this is a closed book exam. I pledge on my honor that for the duration of the exam I shall not consult anyone in this course or outside, any online resources, or any other material provided in the course or by outside sources, on questions relevant to this exam. Nor shall I use any computational resources (including calculators or other programmable computing devices) in answering the exam problems. Also, I shall not help any other student with questions relevant to this exam, until after the exam submission is over. I understand that any violation of this code of conduct will be treated as an instance of cheating.

________________________________ **(signature.)**

Turning in the exam

The exam is due at **11:00 am on Thursday Nov 15**.
Please take a printout of the exam and write your answer in the space provided. To turn it in, you can bring it to the lecture, or drop it off at the dropboxes in Siebel Center, prior to coming to class.

INSTRUCTIONS (read carefully)

- Before starting the exam please read and sign the honor-pledge.
- There are 9 problems (with sub-problems) in this exam. In addition there are two optional extra-credit problems (problems 10 and 11).
- The point value of each problem is indicated next to the problem, and on the cover page table. It is wise to skim all problems and point values first, to best plan your time.
- In problems that ask you to select one or more options, you should place a check mark on the box next to the option(s) that you select, like so:

  ☑ C. My option.

  If you make a mistake and want to change your answers (and can't cleanly erase your original selection), cross out all the boxes, and write the letter(s) for your choice(s) clearly. If your selections are ambiguous, you may lose points.
- Each multiple-choice problem has one or more correct choices. For full credit, you should select all the correct choices and none of the wrong choices.
- When writing proofs, use your best mathematical style, e.g. introduce variables and assumptions, justify important or non-obvious steps, put your steps in logical order.
- Do all work in the space provided, using the backs of sheets if necessary. Brief explanations and/or showing work, even when not explicitly asked for, may increase partial credit for buggy answers. If you have work on the backside that you would like the grader to see, then you must clearly indicate so on the problem. See the proctor if you need more paper.
- Extra credit: Any points you earn from problems 10 and 11 will will simply be added to your total. (Your total will be truncated to a maximum of 100 after that.) The points from the extra credit problem will also be recorded as part of your total extra credits, that would be used to possibly upgrade a final grade (e.g., from A to an A+).
- Points may be deducted for solutions which are correct, but hard to read, hard to understand, poorly explained, or excessively complicated.
- Please bring any apparent bugs or ambiguity to the attention of the instructor/TAs.
- Adhere to the honor-pledge from the front page. After the midterm is over, you may discuss its contents with other students **only** after verifying that they have also taken the exam (e.g. they aren't about to take a makeup exam).

1. **Graphs** [10 points]

In this problem we use an adjacency matrix representation for a graph: an $n \times n$ symmetric matrix M with entries 0 and 1 is used to represent a graph $G = (V, E)$ with vertex set $V = \{1, \ldots, n\}$ and edge-set E, such that $M_{ij} = 1$ if and only if there is an edge $\{i, j\} \in E$.

For example, the cycle graph C_4 with $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}$ has the adjacency matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Consider the graphs G_1, G_2, G_3, G_4 defined below.

$$\underset{G_1}{\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}} \quad \underset{G_2}{\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}} \quad \underset{G_3}{\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}} \quad \underset{G_4}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}$$

(a) The number of edges in G_3 is ________________________.

(b) Select all the statements below that are true. (No justification is needed.)

☐ A. G_1 is isomorphic to K_4

☐ B. G_2 is isomorphic to C_4

☐ C. G_3 is isomorphic to a subgraph of G_2

☐ D. G_4 is isomorphic to a subgraph of C_4

(c) Which of G_1, G_2, G_3, G_4 are bi-partite? ________________________.

(d) Which of G_1, G_2, G_3, G_4 have an Eulerian circuit? ________________________.

2. **Recursion and Trees** [6 points]

Let $T : \mathbb{Z}^+ \to \mathbb{N}$ be as follows (defined for powers of 2):

$$T(1) = c$$
$$T(n) = 2T(n/2) + f(n) \qquad \forall n \in \{2^k | k \in \mathbb{Z}^+\}$$

for some function $f : \mathbb{Z}^+ \to \mathbb{N}$, and some constant $c \in \mathbb{N}$.

(a) Draw a recursion tree for $T(8)$, as defined in class. Fill in all the nodes of the tree, so that $T(8)$ is equal to the sum of the values in the nodes of this tree. Also indicate (in the space below) what this value of $T(8)$ is (in terms of f and c only, without referring to the value of the function T itself).

$T(8) =$ ______________________________ (in terms of f and c only).

(b) Suppose $f(n) = n$ and $c = 0$. Give a closed form expression for $T(n)$ (when n is a power of 2).

3. **Induction.** [6 points]

Below is a proof by (weak) induction, to prove that $\forall m, n \in \mathbb{Z}^+ \ \frac{(m+n)!}{m!n!} \in \mathbb{Z}^+$. The induction variable is $m+n$.

Fill in the six blanks below to complete the proof. You must choose your answers to match the rest of the proof as given here (and not attempt to alter or generalize the rest of the proof). In particular, note that the statement and proof consider only positive integers m, n (though it is possible, using a slightly different proof, to extend this to the set of all natural numbers).

(a) The base case claim is that ____________________________. This claim holds because the only possible values of $m, n \in \mathbb{Z}^+$ in this case are ______________________.

Then, $\frac{(m+n)!}{m!n!} =$ ______________ which is indeed in $\mathbb{Z}^+$.

(b) The induction step. We claim that:

(*range for k:*) $\forall k \geq$ ____, if

(*induction hypothesis:*) $\forall m, n \in \mathbb{Z}^+$ such that $m + n = k$, it holds that $\frac{(m+n)!}{m!n!} \in \mathbb{Z}^+$,

(*to prove:*) then ____.

(c) To prove the above claim, suppose $m, n \in \mathbb{Z}^+$ are arbitrary numbers such that $m + n = k + 1$ where k is as in the claim.

Firstly, we note that for all $m, n \in \mathbb{Z}^+$,

$$\frac{(m+n)!}{m!n!} = \frac{(m+(n-1))!}{m!(n-1)!} + \frac{((m-1)+n)!}{(m-1)!n!}.$$

Now we argue that the first term $\frac{(m+(n-1))!}{m!(n-1)!} \in \mathbb{Z}^+$. For this we consider two cases.

Case $n = 1$: In this case $\frac{(m+(n-1))!}{m!(n-1)!} = \frac{(m+0)!}{m!0!} = 1 \in \mathbb{Z}^+$.

Case $n > 1$: In this case we can apply the (weak) induction hypothesis because $m, (n-1) \in \mathbb{Z}^+$ and ______________________. Then by the induction hypothesis, we get $\frac{(m+(n-1))!}{m!(n-1)!} \in \mathbb{Z}^+$.

Thus in either case $\frac{(m+(n-1))!}{m!(n-1)!} \in \mathbb{Z}^+$, as desired.

Similarly, $\frac{((m-1)+n)!}{(m-1)!n!} \in \mathbb{Z}^+$. Since $\frac{(m+n)!}{m!n!}$ is the the sum of two numbers in $\mathbb{Z}^+$, $\frac{(m+n)!}{m!n!} \in \mathbb{Z}^+$.

4. **Multiple Choice Problems.** [20 points]

This page has 4 problems, each worth 5 points. Each problem has one or more correct choices. **For full credit, you should select all the correct choices and none of the wrong choices.**

I. **Degrees of Graphs.** Choose all the correct statements.

☐ A. If every node in a graph G has degree exactly 1, then G has half as many connected components as it has nodes.

☐ B. If every node in a graph G has degree exactly 2, then G is isomorphic to a cycle graph.

☐ C. If all nodes in G have odd degree, then G must have an even number of nodes.

☐ D. If a graph with n nodes has a node with degree $n - 2$, then either the graph is connected or it has a node of degree 0.

II. **Chromatic Number.** Below, G and H stand for graphs. Choose the correct statements.

☐ A. If G is a proper subgraph of H then $\chi(G) < \chi(H)$.

☐ B. If G is a proper subgraph of K_n then $\chi(G) < n$.

☐ C. If G is isomorphic to H, then $\chi(G) = \chi(H)$.

☐ D. If G has maximum degree n, then $\chi(G) = n + 1$.

III. **Grammar.** Consider the grammar below, with start symbol S.

$$\begin{aligned} S &\to AS \mid SB \mid \epsilon \\ A &\to Aa \mid a \\ B &\to Bb \mid b \end{aligned}$$

Which of the following strings can be generated by this grammar?

☐ A. a

☐ B. abb

☐ C. $abba$

☐ D. $aaabbb$

IV. **Subgraph.** Consider obtaining a subgraph of K_4 by deleting some of its edges (but retaining all the vertices). Which of the following operations is/are guaranteed to create a connected acyclic subgraph (i.e., a graph-theoretic tree)?

☐ A. Delete any 4 edges from K_4 in such a way that the resulting graph is connected.

☐ B. Delete any 2 or 3 edges from K_4 in such a way that the resulting graph is connected.

☐ C. Delete any 3 edges from K_4 in such a way that there are no cycles in the resulting graph.

☐ D. Delete any 3 edges from K_4 in such a way that all 3 deleted edges did not share a common vertex.

5. **Problems with Short Answers.** [9 points]

This page has 3 problems, each worth 3 points. Each problem has subproblems with blanks that you should to fill in with the most appropriate answer. No justification is needed.

(a) **Trees**

i. In a tree with n nodes, the sum of the degrees of all the nodes is ________________ (in terms of n).

ii. Recall that the *arity* of a tree is equal to the maximum number of children any node has in the tree. The maximum possible height of a tree with arity 5 and 31 nodes is ________________, and the minimum possible height of such a tree is ________________.

(b) **Big O.** For each of the following pair of functions $f : \mathbb{N} \to \mathbb{R}^+$ and $g : \mathbb{N} \to \mathbb{R}^+$, give $c, k \geq 0$ to show that $f(n) = O(g(n))$. That is, give $c, k \geq 0$ such that $\forall n \geq k,\ f(n) \leq c \cdot g(n)$. In each case, find the smallest possible *integer* c, and for that value of c, the smallest possible *integer* k. No justification is needed.

i. $f(n) = 10n^2$, $g(n) = \frac{n^2}{2} + 100$. $c =$ ________________, $k =$ ________________.

ii. $f(n) = \frac{n^2}{2} + 100$, $g(n) = 10n^2$. $c =$ ________________, $k =$ ________________.

iii. $f(n) = n^2 + 4$, $g(n) = 2^n$. $c =$ ________________, $k =$ ________________.

(c) **Nested Loops.** Consider the pseudocode below in which the variable `count` keeps track of the number of addition/subtraction operations on the variables `i` and `j`. At the end it prints this count.

```
Loopy(n: int) {
    i := 0
    j := n
    count := 0
    while (i < j) {
        while (j > (i+j)/2 + 4) {
            count := count + 1
            j := j - 1
        }
        count := count + 1
        i := i + 1
    }
    print count
}
```

What is the output printed by the function on input $n \in \mathbb{N}$ (as a function of n)? ________________

[Hint: When the outer-loop exits, it will be the case that `i = j`.]

6. **Running Time of an Algorithm.** [10 points]

Consider the two functions below, each of which computes $x^{(2^d)}$, for some real number x and non-negative integer d.

```
RSq1(x: real, d: int) {
     if (d=0)
          return x
     else
          return(RSq1(x,d-1) * RSq1(x,d-1))
}
```

```
RSq2(x: real, d: int) {
     if (d=0)
          return x
     else {
          y := RSq2(x,d-1)
          return y*y
     }
}
```

Let $T_1(\mathtt{d})$ be the time taken by `RSq1`, on input `(x,d)`, and let $T_2(\mathtt{d})$ the time taken by `RSq2`. (Assume that all operations in the code, except the recursive computation, take time, independent of `d`.)

(a) Give a recurrence relation (including base case) for $T_1(\mathtt{d})$.

(b) Find a closed form expression for $T_1(\mathtt{d})$. It is enough to write it as $\Theta(\cdot)$ expression (using polynomial, logarithmic or exponential functions). No explanation is needed (but may help with partial credit).

(c) Give a recurrence relation (including base case) for $T_2(\mathtt{d})$.

(d) Find a closed form expression for $T_2(\mathtt{d})$. It is enough to write it as $\Theta(\cdot)$ expression (using polynomial, logarithmic or exponential functions). No explanation is needed (but may help with partial credit).

7. **Grammar** [12 points]

Consider the following grammar. The set of terminals is $\{a, b, +, \times, (,)\}$, and the set of variables is $\{\texttt{Exp}, \texttt{Term}, \texttt{Factor}\}$. `Exp` is the start-symbol.

$$\begin{aligned} \texttt{Exp} &\to \texttt{Exp} + \texttt{Term} \mid \texttt{Term} \\ \texttt{Term} &\to \texttt{Term} \times \texttt{Factor} \mid \texttt{Factor} \\ \texttt{Factor} &\to (\texttt{Exp}) \mid a \mid b \end{aligned}$$

Give the parse tree generating the following string:

$$a \times b + b \times (a + b).$$

8. **Proof by Induction.** [12 points]

Prove the following two statements using mathematical induction. Clearly state what your induction variable is, state and prove the base case, and state and prove the induction step. In the induction step, clearly indicate what range of values it applies to, what the induction hypothesis is and what the statement implied by it is.

(a) Prove by induction: For all $h \in \mathbb{N}$, the number of nodes in a full and complete m-ary tree of height h is (for any integer $m > 1$) is

$$\frac{m^{h+1} - 1}{m - 1}.$$

(b) Prove by induction: For all $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} k \cdot 2^k = (n-1) \cdot 2^{n+1} + 2.$$

9. **Write a Grammar.** [15 points]

Give a context free grammar that generates all strings composed of 0s and 1s that have equal number of 0s and 1s.

[Hint: Given a non-empty string with equal number of 0s and 1s, if you scan it from left to right, there are 3 possibilities:

- At all points till you each the last position, the number of 0s encountered is strictly more than the number of 1s. (Then the first position must have a 0 and the last position must have a 1.)
- At all points till you each the last position, the number of 1s encountered is strictly more than the number of 0s. (Then the first position must have a 1 and the last position must have a 0.)
- There is some position before the last position when the number of 0s and number of 1s encountered so far are equal.

In each case, can you break down the string into smaller piece(s) with equal number of 0s and 1s?]

10. **Extra Credit.** [12 points]

This problem relates to adjacency matrices of graphs. The graphs in this problem may have self-loops, but are otherwise simple (i.e., undirected, and no multi-edges).

Consider a boolean matrix multiplication defined as follows: If A is an $\ell \times m$ matrix and B is an $m \times n$ matrix with 0-1 values (identified with boolean values F and T), then $A \cdot B$ is an $\ell \times n$ matrix M defined by

$$M_{ij} = (A_{i1} \wedge B_{1j}) \vee (A_{i2} \wedge B_{2j}) \vee \cdots \vee (A_{im} \wedge B_{mj})$$

or more succicntly $M_{ij} = \bigvee_{k=1}^{m} (A_{ik} \wedge B_{kj})$. That is, boolean matrix multiplication is similar to the standard multiplication, but with logical OR instead of addition and logical AND instead of multiplication. We denote $A \cdot A$ by A^2.

(a) Consider A_n to be the adjacency matrix of the cycle graph C_n. (A_4 is shown in Problem 1.) Compute the matrix A_4^2.

$$A_4^2 =$$

(b) Draw/describe the graph corresponding to A_4^2. How is an edge in this graph related to (paths in) C_4?

(c) In general how is the graph corresponding to A^2 related to the graph corresponding to A? Use this to draw/describe the graph corresponding to A_6^2.

11. **Extra Credit.** [8 points]

Consider the function $g : \mathbb{N} \to \mathbb{N}$ defined as $g(0) = 0$ and, for all $n \in \mathbb{Z}^+$, $g(n) = 2g(n-1) + n$. Find a closed form for g, *without a summation expression.* (Use the expressions from Problem 8 to remove any summation notation from your closed form.) For partial credit, give a $O(\,\cdot\,)$ bound on $g(n)$ that is as tight as you can make it.