Midterm Review

David Holcomb

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1 Grammar

1.1

Consider the following grammar, with set of non-terminals $\{S, T, R, X\}$ and set of terminals $\{a, b, \epsilon\}$. S is the start symbol.

$$\begin{aligned} \texttt{S} &\rightarrow XSX \mid T \\ T &\rightarrow aRb \mid bRa \\ R &\rightarrow XR \mid \epsilon \\ X &\rightarrow a \mid b \end{aligned}$$

This grammar enforces that any string generated has one of the following forms

$$\texttt{X} \ldots \texttt{XaX} \ldots \texttt{XbX} \ldots \texttt{X}$$

$$\texttt{X} \ldots \texttt{XbX} \ldots \texttt{XaX} \ldots \texttt{X}$$

where the there are equal number of Xs (possibly 0) on the "oustide" and some arbitrary number of Xs (possibly 0) in the middle. (Each X becomes a single character a or b.) This ensures that any string generated by this grammar is *not a palindrome*.

1.1.1

Draw parse trees for each of the following strings:

1. ab

```
      S
      |
      T
     /|\
    a R b
      |
      ϵ
```

2. abb

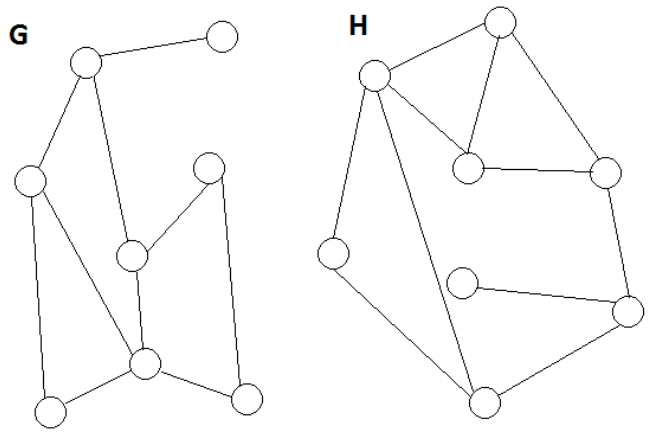
```
      S
      |
      T
     /|\
    a R b
     / \
    X   R
    |   |
    b   ϵ
```

3. ababbba

```
      S
     /|\
    X S X
   / /|\ \
  a X S X a
   /  |  \
  b   T   b
     /|\
    a R b
     / \
    X   R
    |   |
    b   ϵ
```

2 Graphs

2.1



2.1.1

Are G and H connected? **Solution:** Yes, there are paths between every pair of vertices.

2.1.2

What are the maximum degrees of G and H? **Solution:** Both graphs have maximum degree 4. i.e., the largest degree of any vertex in either graph is 4.

2.1.3

What are the chromatic numbers of G and H? Justify your answer. **Solution:** Both graphs have K_3 as a subgraph, so they must use at most 3 colors. But both can be colored with exactly 3 colors.

2.1.4

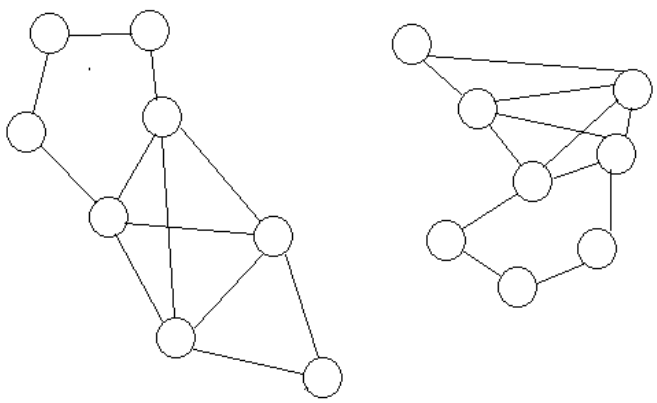
Are G and H isomorphic? Justify your answer. **Solution:** G and H are not isomorphic. G has 3 vertices of degree 2. H has 1 vertex of degree 2. This alone makes it so that there can be no bijection between the vertices that preserves adjacencies.

2.2

If G,H are graphs, here are their respective adjacency matrices:

$$A_G = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, A_H = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Redrawing the graphs, we get something like



2.2.1

Are G and H connected? **Solution:** Yes, there are paths between any pair of vertices.

2.2.2

What are the maximum degrees of G and H? **Solution:** Both graphs have maximum degree 4.

2.2.3

What are the chromatic numbers of the G and H? Justify your answer. **Solution:** The chromatic numbers of both are 4. Both contain K_4 as a

subgraph, so they will need at least 4 colors. We can color both with 4 colors.

2.2.4

Do G and H contain Eulerian circuits? Justify your answer. **Solution:** Yes. Just by looking at the adjacency matrices, we find an even number of 1's on any row or column on either graph.

2.2.5

Are G and H isomorphic? Justify your answer. **Solution:** Yes. This can be verified by examining the graphs.

3 Induction

3.1

Let σ_n denote the sum of the digits of n . For example $\sigma_{37} = 10$, $\sigma_{543} = 12$. Prove that $n \equiv \sigma_n \ (mod\ 9)$ using induction. (Hint: What happens if n has a 9 in the singles digit, and we increase n by 1?)

Solution: Base: $n = 0$. $\sigma_0 = 0$. Therefore $0 \equiv \sigma_0 \ (mod\ 9)$.

IH: Suppose $n \equiv \sigma_n \ (mod\ 9)$ for all $n = 0, 1, 2, ..., k$.

Then, $\sigma_{k+1} = \begin{cases} \sigma_k + 1 \text{ if singles digit of k is not 9} \\ \sigma_k - 9c + 1 \text{ if singles digit of k is 9} \end{cases}$. If the singles digit of k is not 9, the singles digit is the only digit of k that changes when we add 1 to k. If the singles digit of k is 9, when we increase k by 1, we will have to make the singles digit into 0 and carry a 1. This may result in the tens digit going from 9 to 0 and carrying another one. This may result in the hundreds digit going from 9 to 0 and carrying yet another one. This process may continue until we hit a digit of k that isn't 9, which will be increased by 1.

Then, in the first case, because $k \equiv \sigma_k \ (mod\ 9)$, we can say that $k + 1 \equiv \sigma_k + 1 \ (mod\ 9)$ or $k + 1 \equiv \sigma_{k+1} \ (mod\ 9)$.

In the second case, we know that $k - \sigma_k = 9b$. Therefore $k + 1 - \sigma_{k+1} = k + 1 - (\sigma_k - 9c + 1) = k - \sigma_k + 9c. = 9b + 9c$. Therefore $k + 1 \equiv \sigma_{k+1} \ (mod\ 9)$.

3.2

Let the function $f : N \to Z$ be defined by
$f(0) = 1$
$f(1) = 6$
$\forall n \geq 2, f(n) = 6f(n-1) - 9f(n-2)$
Use strong induction on n to prove that $\forall n \geq 0, f(n) = (1+n)3^n$

Solution: Base: $n = 0$. Then $f(0) = 1 = (1+0)3^0$.

IH: Suppose $f(n) = (1+n)3^n$ for all $n = 0, 1, 2, ..., k$.

Then, $f(n+1) = 6f(n) - 9f(n-1) = 6(n+1)3^n - 9(n)3^{n-1} = 2 \cdot 3(n+1)3^n - 3^2(n)3^{n-1} = 2(n+1)3^{n+1} - n \cdot 3^{n+1} = 3^{n+1}(2(n+1) - n) = 3^{n+1}(2n+2-n) = 3^{n+1}(n+2)$.

3.3

3.3.1

Prove that n^0 is $O(n^1)$.

Let $C = 1, k = 1$. Then $n \geq 1 = n^0$.

3.3.2

Prove that n^z is $O(n^{z+1})$ for any $z \in \mathbb{N}$.

Let $C = 1, n = 2$. Then $n^{z+1} \geq n \cdot n^z \geq 2 \cdot n^z \geq n^z$.

4 Recursion Trees

4.1

Find a closed form for
$T(n) = 2T(\frac{n}{3}) + n + 1$
$T(1) = 2$ when n is a power of 3.

Solution: The zeroth level has one node with value $n+1$. The first level has two nodes with value $\frac{n}{3} + 1$ each, which totals to $2\frac{n}{3} + 2$. The second level has four nodes with value $\frac{n}{9} + 1$ each, which totals to $4\frac{n}{9} + 4$. In general, the sum of the nodes at level i is $2^i(\frac{n}{3^i} + 1)$. There are $1 + \log_3 n$ levels in the tree, running from $i = 0$ to $i = \log_3 n$ because, at each level, the n-value is one-third the n-value of the level above, and when n-value is 1, we have

reached the leaves. (Note that if $n = 1 = 3^0$, the leaf level is level 0; if $n = 3^1$, the leaf level is level 1.)

Then, the total for the recursion tree would be $\sum_{i=0}^{\log_3 n} 2^i(\frac{n}{3^i} + 1) = \sum_{i=0}^{\log_3 n} n\left(\frac{2}{3}\right)^i + \sum_{i=0}^{\log_3 n} 2^i$. For convenience, let $m = \log_3 n$. Then summing the first geometric series gives $n\frac{1-(2/3)^m}{1-2/3} = 3n(1-(2/3)^m)$. The second series sums up to $2^{1+\log_3 n} - 1 = 2 \cdot 2^{\log_2 n \times \log_3 2} - 1 = 2 \cdot n^{\log_3 2} - 1$. Thus $T(n) = 3n(1 - \left(\frac{2}{3}\right)^{\log_3 n}) + 2 \cdot n^{\log_3 2} - 1$.

Note: $T(n) = \Theta(n)$.

4.2

Find a closed form for
$T(n) = 2T(n-1) + 3$
$T(1) = 3$

Solution: The zeroth level has one node with value 3. The second level has two nodes, each having a value of 3. The third level has four nodes, each having a value of 3. The sum of the values of the nodes at any level is $3 \cdot 2^i$. The tree has levels $i = 0$ to $i = n-1$. (Check: when $n = 1$, the tree has only level 0; when $n = 2$, the tree has levels 0 and 1.) Then, the value of $T(n)$ is $3 \cdot \sum_{i=0}^{n-1} 2^i = 3 \cdot (2^n - 1)$.

Note: $T(n) = \Theta(2^n)$.

4.3

Find a closed form for
$T(n) = T(\lfloor\sqrt{n}\rfloor) + n$
$T(1) = 2$
when $n = 2^{\left(2^k\right)}$ for some non-negative integer k.

Solution: If $n = 2^{2^k}$, note that $\sqrt{n} = 2^{(2^k)/2} = 2^{2^{k-1}}$ is an integer through out the recursive steps, up to $n = 2^{2^0} = 2$. $T(2) = T(\lfloor\sqrt{2}\rfloor) + 2 = T(1) + 2 = 3$. Thus we can ignore the floor (and take $T(2) = 3$ to be the base-case) if $n = 2^{\left(2^k\right)}$.

Writing n_i for the n-value at level i, we have $n_0 = 2^{2^k}$, $n_1 = 2^{2^{k-1}}$, $\ldots$, $n_i = 2^{2^{k-i}}$, $\ldots$, $n_k = 2^{2^0} = 2$ (which we take as the leaf level). Now, the recursion tree is in fact a unary tree (a chain). So at level i we have one node; its value is $n_i = 2^{2^{k-i}}$, except at level k where the value is $T(2) = 3 = 2^{2^0} + 1$. Thus $T(2^{2^k}) = 2^{2^k} + 2^{2^{k-1}} + \cdots + 2^{2^1} + 2^{2^0} + 1$.

When $n = 2^{2^k}$, $k = \log_2(\log_2 n)$ Then we can write $T(n) = 1 + \sum_{i=0}^{\log_2(\log_2 n)} 2^{2^i}$, which would be the "closed form" we are looking for.

Note: The last term in the above summation is n. This summation is smaller than the summation $n + n/2 + n/4 + \ldots$, which is in turn upper-bounded by $2n$. So $n < T(n) < 2n$, and hence $T(n) = \Theta(n)$.

5 Big O

5.1

Order the following functions in increasing (in terms of Big O) order.

$$n^2,\ n\log_3^2 n,\ \sqrt{n^5 + \log_2 n},\ \log_{10} n,\ \log_5 n,\ 2^{\log_3 n},\ (\sqrt{2})^n$$

Solution:

$$\log_{10} n,\ \log_5 n,\ 2^{\log_3 n},\ n\log_3^2 n,\ n^2,\ \sqrt{n^5 + \log_2 n},\ (\sqrt{2})^n$$

5.2

Find $k \in \mathbb{N}$ such that $n^{\sqrt{2}} \log^2 n$ is $O(n^k)$. Is your bound "tight"?

Solution: Let $k = 2$. Then $n^{\sqrt{2}} \log^2 n$ is $O(n^2)$. The bound is not tight.

5.3

Prove that $\sqrt{x^4 + 6x^2 + 9}$ is $O(x^2)$. Find a $C, k \geq 0$ such that $\sqrt{x^4 + 6x^2 + 9} \leq Cx^2$ for all $x \geq k$.

Solution: Let $C = 2, k = 2$. Then $\sqrt{x^4 + 6x^2 + 9} = \sqrt{(x^2+3)^2} = x^2 + 3$. Then, $2x^2 \geq x^2 + x^2 \geq x^2 + 3$.

5.4

Prove that $n^3 + 7n^2$ is not $O(n^2)$. Given a $C, k \geq 0$, find an $n \geq k$ such that $n^3 + 7n^2 > Cn^2$.

Solution: Let $n = \max\{C - 6, k\}$. Then $n > C - 7$. Then $n + 7 - C > 0$. Then $n^3 + n^2(7 - C) > 0$. Then, $n^3 + 7n^2 > Cn^2$.