1 Grammar

1.1

Consider the following grammar, with set of non-terminals \{S, T, R, X\} and set of terminals \{a, b, \epsilon\}. S is the start symbol.

\[
S \rightarrow XSX \mid T \\
T \rightarrow aRb \mid bRa \\
R \rightarrow XR \mid \epsilon \\
X \rightarrow a \mid b
\]

This grammar enforces that any string generated has one of the following forms

\[
X \ldots XaX \ldots XbX \ldots X \\
X \ldots XbX \ldots XaX \ldots X
\]

where the there are equal number of Xs (possibly 0) on the “outside” and some arbitrary number of Xs (possibly 0) in the middle. (Each X becomes a single character a or b.) This ensures that any string generated by this grammar is not a palindrome.

1.1.1

Draw parse trees for each of the following strings:
1. ab

```
S
| T
/ \|a R b
| ϵ
```

2. abb

```
S
| T
/ \|a R b
/ \|X R
| |b ϵ
```

3. ababbba

```
S
/ \|X S X
/ / \ \|a X S X a
/ | | \b T b
/ / \|a R b
/ \|X R
| |b ϵ
```
2 Graphs

2.1

2.1.1
Are $G$ and $H$ connected? **Solution:** Yes, there are paths between every pair of vertices.

2.1.2
What are the maximum degrees of $G$ and $H$? **Solution:** Both graphs have maximum degree 4. i.e., the largest degree of any vertex in either graph is 4.

2.1.3
What are the chromatic numbers of $G$ and $H$? Justify your answer. **Solution:** Both graphs have $K_3$ as a subgraph, so they must use at most 3 colors. But both can be colored with exactly 3 colors.

2.1.4
Are $G$ and $H$ isomorphic? Justify your answer. **Solution:** $G$ and $H$ are not isomorphic. $G$ has 3 vertices of degree 2. $H$ has 1 vertex of degree 2. This alone makes it so that there can be no bijection between the vertices that preserves adjacencies.
2.2

If $G, H$ are graphs, here are their respective adjacency matrices:

$$A_G = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}, \quad A_H = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0
\end{pmatrix}$$

Redrawing the graphs, we get something like

2.2.1

Are $G$ and $H$ connected? **Solution:** Yes, there are paths between any pair of vertices.

2.2.2

What are the maximum degrees of $G$ and $H$? **Solution:** Both graphs have maximum degree 4.

2.2.3

What are the chromatic numbers of the $G$ and $H$? Justify your answer. **Solution:** The chromatic numbers of both are 4. Both contain $K_4$ as a
subgraph, so they will need at least 4 colors. We can color both with 4 colors.

2.2.4

Do $G$ and $H$ contain Eulerian circuits? Justify your answer. Solution: Yes. Just by looking at the adjacency matrices, we find an even number of 1’s on any row or column on either graph.

2.2.5

Are $G$ and $H$ isomorphic? Justify your answer. Solution: Yes. This can be verified by examining the graphs.

3 Induction

3.1

Let $\sigma_n$ denote the sum of the digits of $n$. For example $\sigma_{37} = 10$, $\sigma_{543} = 12$. Prove that $n \equiv \sigma_n \pmod{9}$ using induction. (Hint: What happens if $n$ has a 9 in the singles digit, and we increase $n$ by 1?)

Solution: Base: $n = 0$. $\sigma_0 = 0$. Therefore $0 \equiv \sigma_0 \pmod{9}$.

IH: Suppose $n \equiv \sigma_n \pmod{9}$ for all $n = 0, 1, 2, ..., k$.

Then, $\sigma_{k+1} = \begin{cases} 
\sigma_k + 1 & \text{if singles digit of } k \text{ is not } 9 \\
\sigma_k - 9c + 1 & \text{if singles digit of } k \text{ is } 9 
\end{cases}$. If the singles digit of $k$ is not 9, the singles digit is the only digit of $k$ that changes when we add 1 to $k$. If the singles digit of $k$ is 9, when we increase $k$ by 1, we will have to make the singles digit into 0 and carry a 1. This may result in the tens digit going from 9 to 0 and carrying another one. This may result in the hundreds digit going from 9 to 0 and carrying yet another one. This process may continue until we hit a digit of $k$ that isn’t 9, which will be increased by 1.

Then, in the first case, because $k \equiv \sigma_k \pmod{9}$, we can say that $k + 1 \equiv \sigma_k + 1 \pmod{9}$ or $k + 1 \equiv \sigma_{k+1} \pmod{9}$.

In the second case, we know that $k - \sigma_k = 9b$. Therefore $k + 1 - \sigma_{k+1} = k + 1 - (\sigma_k - 9c + 1) = k - \sigma_k + 9c. = 9b + 9c$. Therefore $k + 1 \equiv \sigma_{k+1} \pmod{9}$. 
3.2

Let the function \( f : N \to Z \) be defined by
\[
\begin{align*}
  f(0) &= 1 \\
  f(1) &= 6 \\
  \forall n \geq 2, f(n) &= 6f(n-1) - 9f(n-2)
\end{align*}
\]

Use strong induction on \( n \) to prove that \( \forall n \geq 0, f(n) = (1 + n)3^n \)

Solution: Base: \( n = 0 \). Then \( f(0) = 1 = (1 + 0)3^0 \).

IH: Suppose \( f(n) = (1 + n)3^n \) for all \( n = 0, 1, 2, ..., k \).

Then, \( f(n + 1) = 6f(n) - 9f(n - 1) = 6(n + 1)3^n - 9(n)3^{n-1} = 2 \cdot 3(n + 1)3^n - 3^2(n)3^{n-1} = 2(n + 1)3^{n+1} - n \cdot 3^{n+1} = 3^{n+1}(2(n + 1) - n) = 3^{n+1}(2n + 2 - n) = 3^{n+1}(n + 2) \).

3.3

3.3.1

Prove that \( n^0 \) is \( O(n^1) \).

Let \( C = 1, k = 1 \). Then \( n \geq 1 = n^0 \).

3.3.2

Prove that \( n^z \) is \( O(n^{z+1}) \) for any \( z \in N \).

Let \( C = 1, n = 2 \). Then \( n^{z+1} \geq n \cdot n^z \geq 2 \cdot n^z \geq n^z \).

4 Recursion Trees

4.1

Find a closed form for
\[
T(n) = 2T\left(\frac{n}{3}\right) + n + 1
\]
\[
T(1) = 2 \text{ when } n \text{ is a power of } 3.
\]

Solution: The zeroth level has one node with value \( n + 1 \). The first level has two nodes with value \( \frac{n}{3} + 1 \) each, which totals to \( 2 \cdot \frac{n}{3} + 2 \). The second level has four nodes with value \( \frac{n}{3} + 1 \) each, which totals to \( 4 \cdot \frac{n}{9} + 4 \). In general, the sum of the nodes at level \( i \) is \( 2^i(\frac{n}{3^i} + 1) \). There are \( 1 + \log_3 n \) levels in the tree, running from \( i = 0 \) to \( i = \log_3 n \) because, at each level, the \( n \)-value is one-third the \( n \)-value of the level above, and when \( n \)-value is 1, we have
reached the leaves. (Note that if \( n = 1 = 3^0 \), the leaf level is level 0; if \( n = 3^1 \), the leaf level is level 1.)

Then, the total for the recursion tree would be \( \sum_{i=0}^{\log_3 n} 2^i \left( \frac{n}{3^i} + 1 \right) = \sum_{i=0}^{\log_3 n} n \left( \frac{2}{3} \right)^i + \sum_{i=0}^{\log_3 n} 2^i \). For convenience, let \( m = \log_3 n \). Then summing the first geometric series gives \( n \left( \frac{1-(2/3)^m}{1-2/3} \right) = 3n(1-(2/3)^m) \). The second series sums up to \( 2^{1+\log_3 n} - 1 = 2 \cdot 2^{\log_2 n \times \log_3 2} - 1 = 2 \cdot n^{\log_3 2} - 1 \). Thus \( T(n) = 3n(1 - \left( \frac{2}{3} \right)^{\log_3 n}) + 2 \cdot n^{\log_3 2} - 1 \).

Note: \( T(n) = \Theta(n) \).

4.2

Find a closed form for \( T(n) = 2T(n-1) + 3 \)

\( T(1) = 3 \)

**Solution:** The zeroth level has one node with value 3. The second level has two nodes, each having a value of 3. The third level has four nodes, each having a value of 3. The sum of the values of the nodes at any level is \( 3 \cdot 2^i \). The tree has levels \( i = 0 \) to \( i = n - 1 \). (Check: when \( n = 1 \), the tree has only level 0; when \( n = 2 \), the tree has levels 0 and 1.) Then, the value of \( T(n) \) is \( 3 \cdot \sum_{i=0}^{n-1} 2^i = 3 \cdot (2^n - 1) \).

Note: \( T(n) = \Theta(2^n) \).

4.3

Find a closed form for \( T(n) = T(\lfloor \sqrt{n} \rfloor) + n \)

\( T(1) = 2 \)

when \( n = 2^{2^k} \) for some non-negative integer \( k \).

**Solution:** If \( n = 2^{2^k} \), note that \( \sqrt{n} = 2^{(2^k)/2} = 2^{2^{k-1}} \) is an integer throughout the recursive steps, up to \( n = 2^{2^0} = 2 \). \( T(2) = T(\lfloor \sqrt{2} \rfloor) + 2 = T(1) + 2 = 3 \). Thus we can ignore the floor (and take \( T(2) = 3 \) to be the base-case) if \( n = 2^{2^k} \).

Writing \( n_i \) for the \( n \)-value at level \( i \), we have \( n_0 = 2^{2^k} \), \( n_1 = 2^{2^{k-1}} \), \ldots, \( n_i = 2^{2^{k-i}} \), \ldots, \( n_k = 2^{2^0} = 2 \) (which we take as the leaf level). Now, the recursion tree is in fact a unary tree (a chain). So at level \( i \) we have one node; its value is \( n_i = 2^{2^{k-i}} \), except at level \( k \) where the value is \( T(2) = 3 = 2^{2^0} + 1 \). Thus \( T(2^{2^k}) = 2^{2^k} + 2^{2^{k-1}} + \cdots + 2^1 + 2^0 + 1 \).
When \( n = 2^k \), \( k = \log_2 (\log_2 n) \) Then we can write \( T(n) = 1 + \sum_{i=0}^{\log_2 (\log_2 n)} 2^{2^i} \), which would be the “closed form” we are looking for.

Note: The last term in the above summation is \( n \). This summation is smaller than the summation \( n + n/2 + n/4 + \ldots \), which is in turn upper-bounded by \( 2n \). So \( n < T(n) < 2n \), and hence \( T(n) = \Theta(n) \).

5 Big O

5.1
Order the following functions in increasing (in terms of Big O) order.

\[ n^2, n \log_3^2 n, \sqrt{n^5 + \log_2 n}, \log_{10} n, \log_5 n, 2\log_3 n, (\sqrt{2})^n \]

Solution:

\[ \log_{10} n, \log_5 n, 2\log_3 n, n \log_3^2 n, n^2, \sqrt{n^5 + \log_2 n}, (\sqrt{2})^n \]

5.2
Find \( k \in \mathbb{N} \) such that \( n^{\sqrt{2} \log_2 n} \) is \( O(n^k) \). Is your bound ”tight”?

Solution: Let \( k = 2 \). Then \( n^{\sqrt{2} \log_2 n} \) is \( O(n^2) \). The bound is not tight.

5.3
Prove that \( \sqrt{x^4 + 6x^2 + 9} \) is \( O(x^2) \). Find a \( C, k \geq 0 \) such that \( \sqrt{x^4 + 6x^2 + 9} \leq Cx^2 \) for all \( x \geq k \).

Solution: Let \( C = 2, k = 2 \). Then \( \sqrt{x^4 + 6x^2 + 9} = \sqrt{(x^2 + 3)^2} = x^2 + 3 \).

Then, \( 2x^2 \geq x^2 + x^2 \geq x^2 + 3 \).

5.4
Prove that \( n^3 + 7n^2 \) is not \( O(n^2) \). Given a \( C, k \geq 0 \), find an \( n \geq k \) such that \( n^3 + 7n^2 > Cn^2 \).

Solution: Let \( n = \max\{C - 6, k\} \). Then \( n > C - 7 \). Then \( n + 7 - C > 0 \). Then \( n^3 + n^2(7 - C) > 0 \). Then, \( n^3 + 7n^2 > Cn^2 \).