Practice Problems & Solutions
Discrete Structures

CS 173 [B] : Fall 2012

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1. True or false?

   (a) $\forall x, y \in \mathbb{Z}, \exists z \in \mathbb{Z}\ (x \leq z \leq y) \vee (x \geq z \geq y)$.

   Solution:
   True. (For e.g., take $z = y$. Note that it is true that $\forall x, y \in \mathbb{Z}, (x \leq y) \vee (x \geq y)$.)

   (b) $\forall x, y \in \mathbb{Z}, \exists z \in \mathbb{Z}\ (x < z \leq y) \vee (x > z \geq y)$.

   Solution:
   False. (Note that $x < z \leq y \rightarrow x < y$ and $x > x \geq y \rightarrow x > y$. For $x = y$, neither $x < y$ nor $x > y$ would hold.)

2. True or False?

   (a) $(3 \equiv 5 \pmod 5) \rightarrow (3 \equiv 4 \pmod 4)$.

   Solution:
   True. (This is vacuously true, since $3 \equiv 5 \pmod 5$ is false.)

   (b) $(2 \equiv 4 \pmod 3) \leftrightarrow (2 \equiv 6 \pmod 4)$.

   Solution:
   False. (The forward direction of the implication is vacuously true. But the reverse direction is false, since $2 \equiv 6 \pmod 4$ but $2 \not\equiv 4 \pmod 3$.)

3. **Pick the odd one.** Four of the following are equal to each other. Find the one which is not equal to the rest.

   (a) $(A \cap B) - (A \cap C)$
   (b) $A \cap B \cap \overline{C}$
   (c) $A \cap (B - C)$
   (d) $\overline{\overline{A} \cup \overline{B} \cup C}$
   (e) $\overline{A \cup B \cup \overline{C}}$

   Solution:
   (e). We show that the other options are identical to the (b).

   (a) $(A \cap B) - (A \cap C) = (A \cap B) \cap \overline{(A \cap C)} = (A \cap B) \cap (\overline{A} \cup \overline{C}) = (A \cap B \cap \overline{A}) \cup (A \cap B \cap \overline{C}) = \emptyset \cup (A \cap B \cap \overline{C})$
   (c) $A \cap (B - C) = A \cap (B \cap \overline{C})$
   (d) $\overline{\overline{A} \cup \overline{B} \cup C} = A \cap B \cap \overline{C}$
   (e) But $\overline{A \cup B \cup \overline{C}} = \overline{A} \cap \overline{B} \cap C$. This is not equal to $A \cap (B - C)$ in general (for e.g., take A to be the universal set, and C to be empty; then this option is $\emptyset$ but $A \cap (B - C) = B$).

4. $A \cap (A \cup B) =$

(a) $A \cup B$

(b) $A \cap B$

(c) A

(d) B

Solution:
(c). You could reason about this intuitively. You can formally prove it using the logical equivalence $p \wedge (p \vee q) \equiv p$, which in turn you can prove using a truth table. (Then you will use $x \in A$ as p and $x \in B$ as q.)

Another way to show this is to start with the following two relations for any sets A and C: $A \cap C \subseteq A$ and $A \subseteq A \cup C$. By the first inclusion, taking $C = A \cup B$, we have $A \cap (A \cup B) \subseteq A$. For inclusion in the other direction, first we use the distributive property to get $A \cap (A \cup B) = (A \cap A) \cup (A \cap B) = A \cup (A \cap B)$. Now we use the second relation from above, but using $A \cap B$ as C. Then $A \subseteq A \cup (A \cap B)$.

Thus we see that $A \subseteq A \cup (A \cap B) = A \cap (A \cup B) \subseteq A$. Since this set contains A and is contained in A it is equal to A.

5. Let $a = 11^7$. Then (compute by hand – i.e., show your work using small numbers):

   (a) $a \equiv 0 \pmod{12}$

   (b) $a \equiv 1 \pmod{12}$

   (c) $a \equiv 7 \pmod{12}$

   (d) $a \equiv 11 \pmod{12}$

   (e) None of the above.

   Solution:
   $11 \equiv (-1) \pmod{12}$. So $11^7 \equiv (-1)^7 \pmod{12}$. But $(-1)^7 = -1$. So So $11^7 \equiv (-1) \pmod{12}$. Again using the first congruence above, $11^7 \equiv 11 \pmod{12}$.

6. The equivalence classes of integers modulo 2, represented by 0 and 1 can be considered as F and T in propositional calculus. Then modular arithmetic operations can be interpreted as logical operations.

   (a) Write down a table with columns x, y and $x + y$ for addition modulo 2 (where all the values in the table are 0 or 1). Interpreting 0 as the truth value F and 1 as the truth value T, what binary operation does addition modulo 2 correspond to?

   Solution:

x	y	$x+y$
0	0	0
0	1	1
1	0	1
1	1	0

   Substituting F for 0 and T for 1, this gives the truth table of the XOR operator, $\oplus$.

   (b) Write down a table with columns x, y and $x \cdot y$ for multiplication modulo 2. Interpreting 0 as the truth value F and 1 as the truth value T, what binary operation does multiplication modulo 2 correspond to?

Solution:

x	y	$x \cdot y$
0	0	0
0	1	0
1	0	0
1	1	1

Substituting F for 0 and T for 1, this gives the truth table of the AND operator, $\wedge$.

7. The following familiar process in proofs is called *modus ponens*. "If statement 1 holds, then statement 2 holds. Statement 1 holds. Therefore, statement 2 holds." Which of the following logical expressions best captures this? For the statement you choose, prove that it is correct by rewriting the operator $\rightarrow$ in terms of $\neg$ and $\vee$, and manipulations of the logical expressions.

   (a) $(p_1 \wedge p_2) \rightarrow p_2$

   (b) $(p_1 \rightarrow p_2) \rightarrow p_2$

   (c) $(p_1 \wedge (p_1 \rightarrow p_2)) \rightarrow p_2$

   (d) $((p_1 \rightarrow p_2) \rightarrow p_1) \rightarrow p_2$

   Solution:
   (c). Writing p_1 for "Statement 1" and p_2 for "Statement 2," the two statements from which the conclusion drawn can be written as $(p_1 \rightarrow p_2) \wedge p_1$. Drawing the conclusion p_2 is valid because this formula does imply p_2.

   (This can be verified as follows: $(p_1 \rightarrow p_2) \wedge p_1 \equiv (\neg p_1 \vee p_2) \wedge p_1 \equiv (\neg p_1 \wedge p_1) \vee (p_2 \wedge p_1) \equiv F \vee (p_2 \wedge p_1) \equiv (p_2 \wedge p_1)$. But we can see that $(p_2 \wedge p_1) \rightarrow p_2$. (To be pedantic, we can prove the last step as follows: $(p_2 \wedge p_1) \rightarrow p_2 \equiv \neg(p_2 \wedge p_1) \vee p_2 \equiv \neg p_2 \vee \neg p_1 \vee p_2 \equiv T \vee \neg p_2 \equiv T$.))

   (The only other tautology in the list of choices is (a). This appeared in the above argument, but this by itself does not correspond to modus ponens.)

8. Prove $(p \wedge q) \rightarrow p \equiv T$ and $p \rightarrow (p \vee q) \equiv T$. You can either use a truth table (e.g., for the first expression, list columns for p, q, $p \wedge q$ and $(p \wedge q) \rightarrow p$), or manipulate the expressions.

   Solution:

$$(p \wedge r) \rightarrow p \equiv \neg(p \wedge r) \vee p \equiv \neg p \vee \neg r \vee p \equiv (\neg p \vee p) \vee \neg r \equiv T \vee \neg r \equiv T.$$
$$p \rightarrow (p \vee r) \equiv \neg p \vee (p \vee r) \equiv (\neg p \vee p) \vee r \equiv T \vee r \equiv T.$$

9. Given a relation R over the set A (i.e., $R \subseteq A \times A$), we can define a new relation $\overline{R}$ so that for every $x, y \in A$, $(x, y) \in \overline{R}$ iff $(x, y) \notin R$. For each of the following statements, either indicate that it is true or give a counter-example to show that it is false (you can define suitable relations R for your counter-examples):

   (a) if R is reflexive, then $\overline{R}$ is irreflexive.

   Solution:
   True. $(\forall x \in A, (x, x) \in R) \rightarrow (\forall x \in A, (x, x) \notin \overline{R})$.

(b) if R is transitive, then $\overline{R}$ is not transitive.

Solution:
False. For e.g., consider R to be the empty relation, $R = \emptyset$. Then R is transitive, but so is $\overline{R} = A \times A$.

(c) if R is symmetric, then $\overline{R}$ is not symmetric.

Solution:
False. In fact, if R is symmetric, then $\overline{R}$ is symmetric too. As a concrete counter example, we can again consider R to be the empty relation, $R = \emptyset$, so that $\overline{R} = A \times A$.

(d) if R is symmetric, then $\overline{R}$ is anti-symmetric.

Solution:
False. See above example.

10. Suppose $f : A \to B$. Define relation $\sim$ over the set A as follows. $x \sim y$ iff $f(x) = f(y)$. Then prove that $\sim$ is an equivalence relation.

Solution:
<u>Reflexive:</u> For any x, since $f(x) = f(x)$, we have $x \sim x$.

<u>Symmetric:</u> For all x, y, $x \sim y \leftrightarrow f(x) = f(y) \leftrightarrow f(y) = f(x) \leftrightarrow y \sim x$.

<u>Transitive:</u> Suppose x, y, z are such that $x \sim y$ and $y \sim z$. That is, $f(x) = f(y)$ and $f(y) = f(z)$. Hence $f(x) = f(z)$ and therefore $x \sim z$.

11. "Morse Code" is a function M that maps the characters in the English alphabet (only upper case) to sequences of dots and dashes. (It can also handle digits and punctuation, but we'll ignore that.) Each sequence could be of length 1 to 4. We consider them as ordered tuples (containing one, two, three or four elements). Thus for e.g., since the character "A" is mapped to the sequence "• ▬ ", we have $M(A) = (\bullet, \blacksquare)$.

(a) What are the sizes of the domain and co-domain of M as described above?

Solution:
The domain is the English alphabet, and hence its size is 26. The co-domain consists of binary (• and ▬) sequences of lengths 1, 2, 3 and 4. Its size is therefore $2^1 + 2^2 + 2^3 + 2^4 = 30$.

(b) Just knowing the domain and co-domain, and the fact that the Morse code was used to transmit messages error-free over telegraph systems, what can you say about M: is it onto, one-to-one or bijective? Answer each one as "yes, no or can't say."

Solution:
<u>Onto.</u> No, M is not onto, since the co-domain is larger than the domain.

<u>One-to-one.</u> Yes, since decoding a message in Morse code back into English requires that at most one character is mapped to a sequence in Morse code.

<u>Bijective.</u> No, since M is not onto.

12. Definitions:

(a) For integers a, b, define the relation $a|b$.

Solution:
$a|b$ iff $\exists c \in \mathbb{Z}, b = ac$.

(b) For integers a, b, m, define the relation $a \equiv b \pmod{m}$ (in terms of the "divides" relation).

Solution:
$a \equiv b \pmod{m}$ iff $m|(a-b)$.

13. In how many ways can you put 7 pigeons into 8 pigeonholes if no two pigeons can be in the same pigeonhole?

Solution:
8!. This is the number of one-to-one functions from the set of pigeons to the set of pigeonholes. There are $\frac{8!}{1!} = 8!$ such functions.

14. Given a finite set A, with $|A| = n$, how many distinct symmetric relations over $A \times A$ exist?

[Hint: Consider filling up the $n \times n$ matrix denoting the relation, with binary values. Since the relation must be symmetric, it gets fully specified once say, the "lower triangular" part of the matrix and the diagonal are filled up.]

Solution:
$2^{\frac{n(n+1)}{2}}$. There are $(n^2 - n)/2$ unordered pairs of the form $\{a, b\}$ where $a \neq b$, and n pairs of the form (a, a). The relation can choose to include or exclude each of these pairs. (When an unordered pair $\{a, b\}$ is included in the relation, both (a, b) and (b, a) are included; similarly for exclusion). Thus the relation has to choose a subset of $(n^2 - n)/2 + n = n(n+1)/2$ unorderd pairs. This can be done in $2^{\frac{n(n+1)}{2}}$ ways.

In terms of the matrix representation of the relation, a symmetric relation can be specified by the lower triangle and the diagonal entries of the matrix. There are $(n^2 - n)/2$ cells in the lower triangle, and n cells in the diagonal, together making $(n^2 + n)/2$ cells to be specified. Since each of these cells can be included in the relation ("turned on") or not, there are $2^{(n^2+n)/2}$ ways of doing this.