Problem 1: Checkbox (10 points)

Check the box that best characterizes each item.

A path is a type of

- open walk [✓]
- closed walk [ ]
- neither [ ]

Is a graph with a single vertex connected?

- yes [✓]
- no [ ]

If $D$ is the maximum vertex degree in a graph $G$, then the chromatic number of $D$ is

- $= D$ [ ]
- $= D + 1$ [ ]
- $\leq D + 1$ [✓]

The complete bipartite graph $K_{5,7}$ has 35 edges.

- true [✓]
- false [ ]

A loop edge cannot occur in a simple graph.

- true [✓]
- false [ ]

A bipartite graph is a graph with two connected components.

- true [ ]
- false [✓]

If a graph $G$ contains a copy of $K_4$, then the chromatic number of $G$ must be

- $= 4$ [ ]
- $\geq 4$ [✓]
- $\leq 4$ [ ]

The Handshaking Theorem relates the number of edges in a graph to the degrees of its vertices.

- yes [✓]
- no [ ]

A graph has an Euler circuit if and only if at least two vertices have even degree.

- true [ ]
- false [✓]
Problem 2: Counting (9 points)

(a) (4 points) How many different 10-letter strings can be made by rearranging the characters in the word “minimalist”? Show your work.

Solution: \( \frac{10!}{2! \cdot 3!} \) possible strings. This string contains two m’s and three i’s, so the naive answer (10!) needs to be corrected for overcounting (2! for the m’s and 3! for the i’s).

Note: It’s ok, but not necessary, to simplify this expression by cancelling factors.

(b) (4 points) How many different 9-letter strings can be made by rearranging the characters in the word “animation”? Show your work.

Solution: \( \frac{9!}{2! \cdot 2! \cdot 2!} \) possible strings. This string contains two copies each of n, m and i. So the naive answer (9!) needs to be divided by 2! for each of these three letters.

(c) (5 points) How many isomorphisms are there from the graph \( W_6 \) (shown below) to itself? Explain why your answer is correct.

```
     a          b
    / \        / \       / \       / \       / \       / \       / \       / \       / \   
   c     h     d     e     f
```

Solution: The vertex h must map to itself, because it’s the only vertex with degree 6. The vertex a could be mapped to any of the other 6 vertices. However, once a is chosen, we have only two choices for the image of b and then exactly one choice for each of the remaining vertices. So there are 12 isomorphisms.

(d) (5 points) How many isomorphisms are there from the graph \( W_5 \) (shown below) to itself? Explain why your answer is correct.

```
     a
    / \           / \     
 b   h   c     d   e
```

Solution: The analysis is just like the analysis for \( W_6 \), except that there are only 5 choices for the image of a and, therefore, only 10 isomorphisms.
Problem 3: Short proof (10 points)

(a) (5 points) Let’s define \( f : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2 \) such that \( f(x, y) = (x + y, x - y + 3) \). Prove that \( f \) is one-to-one.

**Solution:** Let \((x, y)\) and \((p, q)\) be elements of \(\mathbb{Z}^2\) and suppose that \( f(x, y) = f(p, q) \). Then \((x + y, x - y + 3) = (p + q, p - q + 3)\). So \(x + y = p + q\) and \(x - y + 3 = p - q + 3\). Adding these two equations, we get \(2x + 3 = 2p + 3\), so \(2x = 2p\), so \(x = p\). Substituting \(x = p\) into \(x + y = p + q\), we find that \(p + y = p + q\), so \(y = q\). Since \(x = p\) and \(y = q\), \((x, y) = (p, q)\), which is what we needed to show.

(b) (5 points) Let’s define \( g : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2 \) such that \( g(x, y) = (x - 2y, x + y) \). Prove that \( g \) is one-to-one.

**Solution:** Let \((x, y)\) and \((p, q)\) be elements of \(\mathbb{Z}^2\) and suppose that \( g(x, y) = g(p, q) \). Then \((x - 2y, x + y) = (p - 2q, p + q)\). So \(x - 2y = p - 2q\) and \(x + y = p + q\). Subtracting these two equations, we get \(3y = 3q\), so \(y = q\). Substituting \(y = q\) into \(x - 2y = p - 2q\), we find that \(x - 2y = p - 2y\), so \(x = p\). Since \(x = p\) and \(y = q\), \((x, y) = (p, q)\), which is what we needed to show.

(c) (5 points) Let’s define a relation \( R \) on \(\mathbb{Z}^2\) such that \((x, y)R(p, q)\) if and only if \(x \leq p\) and \(q \leq y\). Prove that \( R \) is antisymmetric.

**Solution:** Let \((x, y)\) and \((p, q)\) be elements of \(\mathbb{Z}^2\). Suppose that \((x, y)R(p, q)\) and \((p, q)R(x, y)\). By the definition of \( R \), \((x, y)R(p, q)\) implies that \(x \leq p\) and \(q \leq y\). Similarly \((p, q)R(x, y)\) implies that \(p \leq x\) and \(y \leq q\). Since \(x \leq p\) and \(p \leq x\), \(x = p\). Since \(y \leq q\) and \(q \leq y\), \(y = q\). So \((x, y) = (p, q)\), which is what we needed to show.

(d) (5 points) Let’s define a relation \( T \) on \(\mathbb{Z}^2\) such that \((p, q)T(a, b)\) if and only if \(p \geq a\) and \(q \geq b\). Prove that \( T \) is antisymmetric.

**Solution:** Let \((a, b)\) and \((p, q)\) be elements of \(\mathbb{Z}^2\). Suppose that \((a, b)T(p, q)\) and \((p, q)T(a, b)\). By the definition of \( T \), \((a, b)T(p, q)\) implies that \(p \geq a\) and \(q \geq b\). Similarly \((p, q)T(a, b)\) implies that \(a \geq p\) and \(b \geq q\). Since \(a \geq p\) and \(p \geq a\), \(a = p\). Similarly, since \(q \geq b\) and \(b \geq q\), \(b = q\). So \((a, b) = (p, q)\), which is what we needed to show.

Problem 4: Graphs (9 points)

(a) (3 points) Draw a picture of the 4-dimensional hypercube \( Q_4 \) (left). Or, in the other exam version, the complete bipartite graph \( K_{3,4} \) (right).

![Diagram of 4-dimensional hypercube and complete bipartite graph]

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3
Recall that if $G$ is a graph, then $\chi(G)$ is its chromatic number. Suppose that $G$ is a connected graph and $H$ is another connected graph (not connected to $G$). Let’s create a new graph $T$ which consists of a copy of $G$, a copy of $H$, and a new edge that connects a vertex of $G$ to a vertex of $H$. For example, if $G$ is $C_3$ and $H$ is $K_4$, then $T$ would look as follows (where $g$ marks vertices of $G$ and $h$ marks vertices of $H$).

Describe how $\chi(T)$ is related to $\chi(G)$ and $\chi(H)$, justifying your answer.

**Solution:**

$\chi(T)$ is the maximum of $\chi(G)$, $\chi(H)$, and 2.

If either $G$ or $H$ requires $k$ colors, there is no hope of coloring $T$ with fewer than $k$ colors. So let $k$ be the larger of $\chi(G)$ and $\chi(H)$.

**Case 1:** $k = 1$. This happens only when both $G$ and $H$ are tiny graphs with just one vertex. In this case, $T$ consists of two vertices joined by an edge and $\chi(T)$ is clearly 2.

**Case 2:** $k \geq 2$. Suppose that $p$ and $q$ are the vertices of $G$ and $H$ (respectively) that are connected by the new edge. Color $p$ and $q$ with two different colors. Looking just at $G$, finish coloring it with $k$ colors. (We know we can do this, because $\chi(G) \leq k$.) Similarly, looking just at $H$, finish coloring it with $k$ colors. We don’t have to worry about what colors were already assigned to vertices of $G$, because the only connection between the two graphs is the edge from $p$ to $q$ and we’ve already ensured these two vertices have different colors.

We’ve now colored $T$ with $\max(k, 2)$ colors.

**Another version of the construction:** Another way to approach the solution is to assume that $G$ and $H$ are already colored, each with no more than $k$ colors. Now consider the vertices $p$ and $q$ joined by the new edge.

If $p$ and $q$ already have different colors, we are done. If $k$ is 1, then we need to switch one of these vertices to an additional color.

Suppose $k \geq 2$ and $p$ and $q$ have the same color $c$. Pick another color $d \neq c$. In the $G$ part of the graph, swap $c$ for $d$ and vice versa on all the nodes. We now have a coloring for the whole graph $T$. 

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4
Problem 5: Induction (12 points)

Let function $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$$
\begin{align*}
f(0) &= 2 \\
f(1) &= 7 \\
f(n) &= f(n-1) + 2f(n-2), \text{ for } n \geq 2
\end{align*}
$$

Use strong induction on $n$ to prove that $f(n) = 3 \cdot 2^n + (-1)^{n+1}$ for any natural number $n$.

**Solution:** Base case(s): For $n = 0$, we have $3 \cdot 2^0 + (-1)^1 = 3 - 1 = 2$ which is equal to $f(0)$.

For $n = 1$, we have $3 \cdot 2^1 + (-1)^2 = 6 + 1 = 7$ which is equal to $f(1)$.

Inductive hypothesis [Be specific, don’t just refer to “the claim”]: Suppose that $f(n) = 3 \cdot 2^n + (-1)^{n+1}$, for $n = 0, 1, \ldots, k-1$ where $k \geq 2$.

Rest of the inductive step:

$$
\begin{align*}
f(k) &= f(k-1) + 2f(k-2) \quad \text{by definition of } f \\
&= (3 \cdot 2^{k-1} + (-1)^k) + 2(3 \cdot 2^{k-2} + (-1)^{k-1}) \quad \text{by inductive hypothesis} \\
&= (3 \cdot 2^{k-1} + (-1)^k) + 3 \cdot 2^{k-1} + 2(-1)^{k-1} \\
&= 6 \cdot 2^{k-1} + (-1)^k - 2(-1)^k \\
&= 3 \cdot 2^k - (-1)^k \\
&= 3 \cdot 2^k(-1)^{k+1}
\end{align*}
$$

So $f(k) = 3 \cdot 2^k(-1)^{k+1}$, which is what we needed to show.
Problem 5: Induction (12 points)

Let function \( f : \mathbb{N} \rightarrow \mathbb{Z} \) be defined by

\[
\begin{align*}
    f(0) &= 3, \\
    f(1) &= 9, \\
    f(n) &= f(n-1) + 2f(n-2), \text{ for } n \geq 2
\end{align*}
\]

Use strong induction on \( n \) to prove that \( f(n) = 4 \cdot 2^n + (-1)^{n-1} \) for any natural number \( n \).

Solution:

Base case(s): For \( n = 0 \), we have \( 4 \cdot 2^0 + (-1)^{-1} = 4 - 1 = 3 \) which is equal to \( f(0) \).

For \( n = 1 \), we have \( 4 \cdot 2^1 + (-1)^0 = 8 + 1 = 9 \) which is equal to \( f(1) \).

Inductive hypothesis [Be specific, don’t just refer to “the claim”]: Suppose that \( f(n) = 4 \cdot 2^n + (-1)^{n-1} \) for \( n = 0, 1, \ldots, k - 1 \) where \( k \geq 2 \).

Rest of the inductive step:

\[
\begin{align*}
    f(k) &= f(k-1) + 2f(k-2) \\
        &= (4 \cdot 2^{k-1} + (-1)^{k-2}) + 2(4 \cdot 2^{k-2} + (-1)^{k-3}) \quad \text{by definition of } f \\
        &= (4 \cdot 2^{k-1} + (-1)^{k-2}) + 4 \cdot 2^{k-1} + 2(-1)^{k-3} \quad \text{by inductive hypothesis} \\
        &= 8 \cdot 2^{k-1} + (-1)^{k-2} - 2(-1)^{k-2} \\
        &= 4 \cdot 2^k - (-1)^{k-2} \\
        &= 4 \cdot 2^k + (-1)^{k-1}
\end{align*}
\]

So \( f(k) = 4 \cdot 2^k(-1)^{k-1} \), which is what we needed to show.