Planar Graphs II

Margaret M. Fleck

7 December 2008

This lecture continues the discussion of planar graphs (section 9.7 of Rosen). It's a half-lecture due to the makeup quiz.

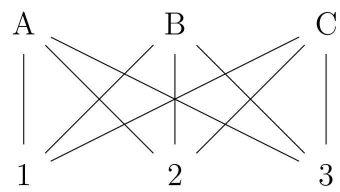
1 Announcements etc

The final exam will be 8-11 am on Thursday December 17th. Main room will be 1404. We also have two overflow rooms, nearby in Siebel. Check your exam schedules for conflicts **RIGHT NOW** because you need to inform instructors by the last day of classes.

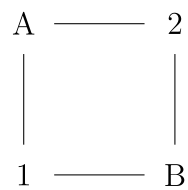
Quiz 3's will be returned in discussion sections today/tomorrow.

2 $K_{3,3}$ isn't planar

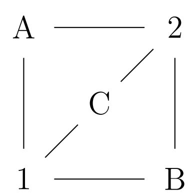
Last lecture, I claimed that $K3, 3$ isn't planar. Let's prove this more carefully. First, let's label the vertices:



The four vertices A, B, 1, and 2 form a cycle.



So C must live inside the cycle or outside the cycle. Let's suppose it lives inside. (The argument is similar if it lives outside.) Our partial graph then looks like:



The final vertex 3 must go into one of the three regions in this diagram. And it's supposed to connect to A, B, and C. But none of the three regions has all three of these vertices on its boundary. So we can't add C and its connections without a crossing.

This proof is ok, but it requires some care to make it convincing. Moreover, it's not going to generalize easily to more complex examples. So let's look for algebraic approaches to establishing that certain graphs aren't planar.

3 Recap

Recall that a simple graph contains no self-loops or multi-edges. For a simple planar graph, remember that we know the following (where e is the number of edges, v is the number of vertices, and f is the number of faces/regions).

- Euler's formula says that $v - e + f = 2$.
- Handshaking theorm: sum of vertex degrees is $2e$
- Second handshaking theorm (planar graphs only): sum of the face degrees is also $2e$.

We can combine these facts to produce some equations that constrain the shape of planar graphs. These equations can be used to quickly prove that certain graphs cannot be planar.

4 A corollary of Euler's formula

Suppose G is a connected simple planar graph, with v vertices, e edges, and f faces, where $f \geq 3$. Then $e \leq 3v - 6$.

> Proof: The sum of the degrees of the regions is equal to twice the number of edges. But each region must have degree ≥ 3. So we have $2e \geq 3f$. Then $\frac{2}{3}e \geq f$.
>
> Euler's formula says that $v-e+f = 2$, so $f = e-v+2$. Combining this with $\frac{2}{3}e \geq f$, we get
>
> $$e - v + 2 \leq \frac{2}{3}e$$
>
> So $\frac{e}{3} - v + 2 \geq 0$. So $\frac{e}{3} \leq v - 2$. Therefore $e \leq 3v - 6$.

From this fact, we can deduce that if G is a connected simple planar graph, then G has a vertex of degree no more than five.

> Proof: This is clearly true if G has one or two vertices.
>
> If G has three vertices, we know that $e \leq 3v - 6$. So $2e \leq 6v - 12$.
>
> By the handshaking theorem, $2e$ is the sum of the degrees of the vertices. Suppose that the degree of each vertex was at least 6. Then we would have $2e \geq 6v$. But this contradicts the fact that $2e \leq 6v - 12$.

We can also use this formula to show that the graph K_5 isn't planar. K_5 has five vertices and 10 edges. This isn't consistent with the formula $e \leq 3v - 6$. Unfortunately, this trick doesn't work for $K_{3,3}$, which isn't planar but satisfies the equation (with 6 vertices and 9 edges).

5 Another corollary

In a similar way, we can show that if G is a connected planar simple graph with e edges and v vertices, with $v \geq 3$, and if G has no circuits of length 3, then $e \leq 2v - 4$.

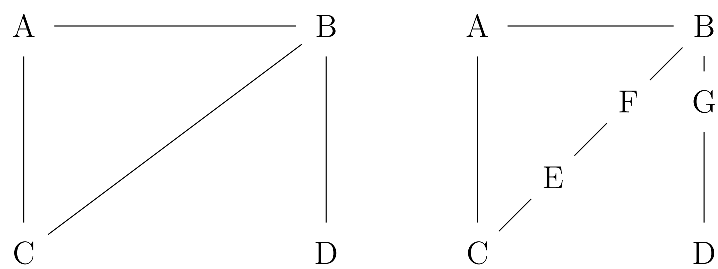
> Proof: The sum of the degrees of the regions is equal to twice the number of edges. But each region must have degree ≥ 4 because we have no circuits of length 3. So we have $2e \geq 4f$. Then $\frac{1}{2}e \geq f$.
>
> Euler's formula says that $v - e + f = 2$. or $e - v + 2 = f$. Combining this with $\frac{1}{2}e \geq f$, we get
>
> $$e - v + 2 \leq \frac{1}{2}e$$
>
> So $\frac{e}{2} - v + 2 \leq 0$. So $\frac{e}{2} \leq v - 2$. Therefore $e \leq 2v - 4$.

We can use this formula to show that $K_{3,3}$ isn't planar.

6 Kuratowski's Theorem

The two example non-planar graphs $K3, 3$ and K_5 weren't picked randomly. It turns out that any non-planar graph must contain a copy of one of these two graphs. Or, sort-of. The copy of $K3, 3$ and K_5 doesn't actually have exactly the literal vertex and edge structure of one of those graphs (i.e. be isomorphic). We need to define a looser notion of graph equivalence, called *homeomorphism*.

A graph G is a *subdivision* of another graph F if G is just like F except that you've divided up some of F's edges by adding vertices in the middle of them. For example, in the following picture, the righthand graph is a subdivision of the lefthand graph.



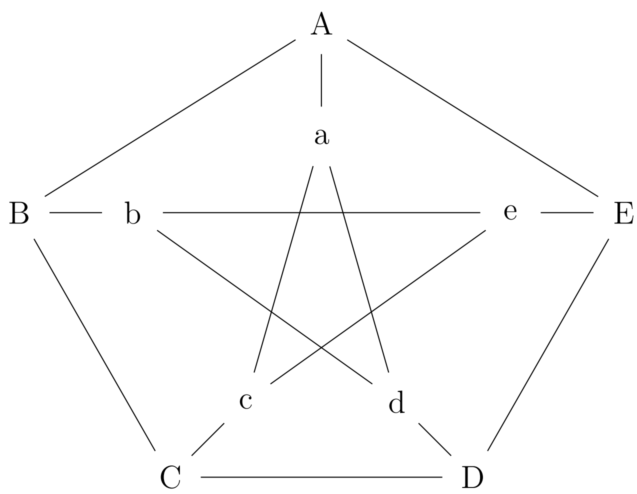
Two graphs are *homeomorphic* if one is a subdivision of another, or they are both subdivisions of some third graph. Graph homeomorphism is a special case of a very general concept from topology: two objects are homeomorphic if you can set up a bijection between their points which is continuous in both directions. For surfaces (e.g. a rubber ball), it means that you can stretch or deform parts of the surface, but not cut holes in it or paste bits of it together.

We can now state our theorem precisely.

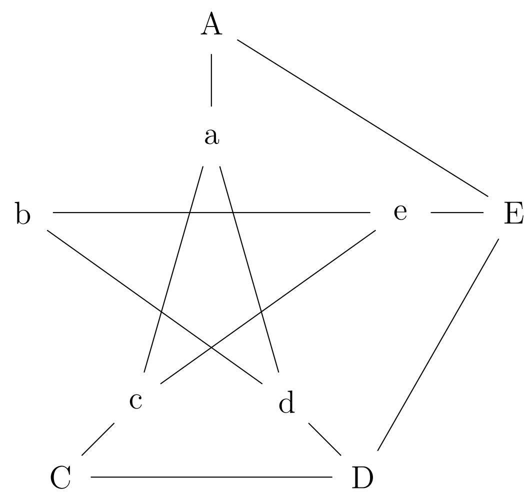
Claim 1 *Kuratowski's Theorem: A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.*

This was proved in 1930 by Kazimierz Kuratowski, and the proof is apparently somewhat difficult. So we'll just see how to apply it.

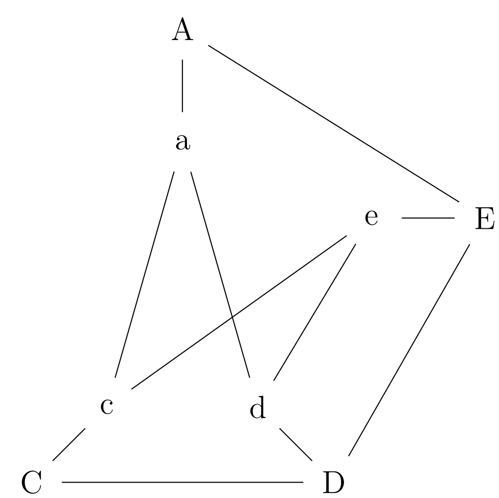
For example, here's a graph known as the Petersen graph (after a Danish mathematician named Julius Petersen).



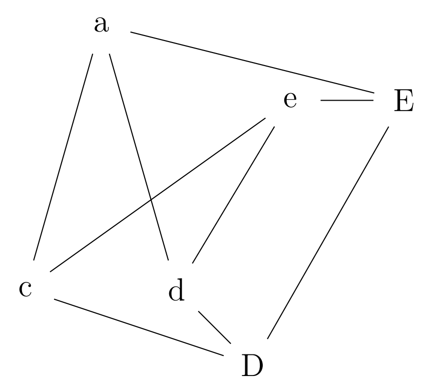
This isn't planar. The offending subgraph is the whole graph, except for the node B (and the edges that connect to B):



This subgraph is homeomorphic to $K_{3,3}$. To see why, first notice that the node b is just subdividing the edge from d to e, so we can delete it. Or, formally, the previous graph is a subdivision of this graph:



In the same way, we can remove the nodes A and C, to eliminate unnecessary subdivisions:



Now deform the picture a bit and we see that we have $K_{3,3}$.