Probability Axioms, Conditional Probability, Statistical (In)dependence, Circuit Problems
**SOLUTION** Because each person can celebrate his or her birthday on any one of 365 days, there are a total of \((365)^n\) possible outcomes. (We are ignoring the possibility of someone having been born on February 29.) Furthermore, there are \((365)(364)(363) \cdots (365 - n + 1)\) possible outcomes that result in no two of the people having the same birthday. This is so because the first person could have any one of 365 birthdays, the next person any of the remaining 364 days, the next any of the remaining 363, and so on. Hence, assuming that each outcome is equally likely, we see that the desired probability is

\[
\frac{(365)(364)(363) \cdots (365 - n + 1)}{(365)^n}
\]

It is a rather surprising fact that when \(n \geq 23\), this probability is less than \(\frac{1}{2}\). That is, if there are 23 or more people in a room, then the probability that at least two of them have the same birthday exceeds \(\frac{1}{2}\). Many people are initially surprised by this result, since 23 seems so small in relation to 365, the number of days of the year. However, every pair of individuals has probability \(\frac{365}{(365)^2} = \frac{1}{365}\) of having the same birthday, and in a group of 23 people there are \(\binom{23}{2} = 253\) different pairs of individuals. Looked at this way, the result no longer seems so surprising.
Let’s check it for our class of ~30 students

Each table, please write your birthdays on the nearest whiteboard in order
(January - top to December – bottom)
Axioms of probability

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:

If $S$ is the sample space and $E$ is any event in a random experiment,

(1) $P(S) = 1$

(2) $0 \leq P(E) \leq 1$

(3) For two events $E_1$ and $E_2$ with $E_1 \cap E_2 = \emptyset$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

These axioms imply that:

$$P(\emptyset) = 0$$

$$P(E') = 1 - P(E)$$

if the event $E_1$ is contained in the event $E_2$

$$P(E_1) \leq P(E_2)$$
Addition rules

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]  \hspace{1cm} (2-1)

If \( A \) and \( B \) are mutually exclusive events,

\[ P(A \cup B) = P(A) + P(B) \]  \hspace{1cm} (2-2)

\[ P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C). \]
The **conditional probability** of an event $B$ given an event $A$, denoted as $P(B \mid A)$, is

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

for $P(A) > 0$.

This definition can be understood in a special case in which all outcomes of a random experiment are equally likely. If there are $n$ total outcomes,

$$P(A) = \frac{\text{(number of outcomes in } A)}{n}$$

Also,

$$P(A \cap B) = \frac{\text{(number of outcomes in } A \cap B)}{n}$$

Consequently,

$$P(A \cap B)/P(A) = \frac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } A}$$

Therefore, $P(B \mid A)$ can be interpreted as the relative frequency of event $B$ among the trials that produce an outcome in event $A$. 
Multiplication rule is just definition of conditional probability

\[ P(B \mid A) = P(B \cap A)/P(A) \to \]

\[ P(B \cap A) = P(B \mid A) \cdot P(A) \]
Drake equation

\[ N = R^* \cdot f_p \cdot n_e \cdot f_l \cdot f_i \cdot f_c \cdot L \]

- \( N \) = The number of civilizations in The Milky Way Galaxy whose electromagnetic emissions are detectable.
- \( R^* \) = The rate of formation of stars suitable for the development of intelligent life.
- \( f_p \) = The fraction of those stars with planetary systems.
- \( n_e \) = The number of planets, per solar system, with an environment suitable for life.
- \( f_l \) = The fraction of suitable planets on which life actually appears.
- \( f_i \) = The fraction of life bearing planets on which intelligent life emerges.
- \( f_c \) = The fraction of civilizations that develop a technology that releases detectable signs of their existence into space.
- \( L \) = The length of time such civilizations release them
Ms. Perez figures that there is a 30% chance that her company will set up a branch in Phoenix. If it does, she is 60% certain that she will be made its manager. What is the probability that Perez will be a Phoenix branch office manager?

A. 90%
B. 18%
C. 30%
D. 60%

Get your i-clickers
Statistically independent events

Always true: \( P(A \cap B) = P(A | B) \cdot P(B) = P(B | A) \cdot P(A) \)

- **Two events**

  Two events are **independent** if any one of the following equivalent statements is true:

  1. \( P(A | B) = P(A) \)
  2. \( P(B | A) = P(B) \)
  3. \( P(A \cap B) = P(A)P(B) \)

- **Multiple events**

  The events \( E_1, E_2, \ldots, E_n \) are independent if and only if for any subset of these events \( E_{i_1}, E_{i_2}, \ldots, E_{i_k} \),

  \[
  P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) = P(E_{i_1}) \times P(E_{i_2}) \times \cdots \times P(E_{i_k})
  \]
Example 3.10. Let an experiment consist of drawing a card at random from a standard deck of 52 playing cards. Define events $A$ and $B$ as "the card is a $\spadesuit$" and "the card is a queen." Are the events $A$ and $B$ independent? By definition, $P(A \cdot B) = P(Q \spadesuit) = \frac{1}{52}$. This is the product of $P(\spadesuit) = \frac{13}{52}$ and $P(Q) = \frac{4}{52}$, and events $A$ and $B$ in question are independent. In this situation, intuition provides no help. Now, pretend that the $2\heartsuit$ is drawn and excluded from the deck prior to the experiment. Events $A$ and $B$ become dependent since

$$P(A) \cdot P(B) = \frac{13}{51} \cdot \frac{4}{51} \neq \frac{1}{51} = P(A \cdot B).$$
Series Circuit

This circuit operates only if there is at least one path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that the devices fail independently. What is the probability that the circuit operates?

Let L & R denote the events that the left and right devices operate. The probability that the circuit operates is:

\[ P(L \text{ and } R) = P(L \cap R) = P(L) \times P(R) = 0.8 \times 0.9 = 0.72. \]
Parallel Circuit

This circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown. Each device fails independently.

Let T & B denote the events that the top and bottom devices operate. The probability that the circuit operates is:

\[ P(T \cup B) = 1 - P(T' \cap B') = 1 - P(T') \cdot P(B') = 1 - 0.05^2 = 1 - 0.0025 = 0.9975. \]
Duality between parallel and series circuits

\[ q_i = 1 - p_i. \]

(a) \hspace{2cm} (b)

<table>
<thead>
<tr>
<th>Connection</th>
<th>Notation</th>
<th>Works with prob</th>
<th>Fails with prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Serial</td>
<td>( E_1 \cap E_2 \cap \cdots \cap E_n )</td>
<td>( p_1 p_2 \cdots p_n )</td>
<td>( 1 - p_1 p_2 \cdots p_n )</td>
</tr>
<tr>
<td>Parallel</td>
<td>( E_1 \cup E_2 \cup \cdots \cup E_n )</td>
<td>( 1 - q_1 q_2 \cdots q_n )</td>
<td>( q_1 q_2 \cdots q_n )</td>
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Advanced Circuit

This circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown. Each device fails independently.

Partition the graph into 3 columns with L & M denoting the left & middle columns.
P(L) = 1 - 0.1^3, and P(M) = 1 - 0.05^2, so the probability that the circuit operates is: (1 – 0.1^3)(1-0.05^2)(0.99) = 0.9875 (this is a series of parallel circuits).
Circuit $\rightarrow$ Set equation
Circuit $\rightarrow$ Set equation

$$P(\text{Works}) = 0.9 \times (1 - (1 - 0.5 \times 0.3) \times (1 - 0.1 \times (1 - 0.6 \times 0.5))) \times 0.8 = 0.15084$$

<table>
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<tr>
<th>Component</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$E_3$</th>
<th>$E_4$</th>
<th>$E_5$</th>
<th>$E_6$</th>
<th>$E_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability of functioning well</td>
<td>0.9</td>
<td>0.5</td>
<td>0.3</td>
<td>0.1</td>
<td>0.4</td>
<td>0.5</td>
<td>0.8</td>
</tr>
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$$E_1 \cap [(E_2 \cap E_3) \cup (E_4 \cap (E_5 \cup E_6))] \cap E_7.$$
Matlab group exercise

• Test our result for this circuit.
• Use circuit_template.m on the website
Matlab check

• Stats=1e6;
• count= 0;
• for i = 1: Stats
• e1 = rand < 0.9; e2 = rand < 0.5; e3 = rand < 0.3;
• e4 = rand < 0.1; e5 = rand < 0.4; e6 = rand < 0.5;
• e7 = rand < 0.8;
• s1 = min(e2,e3); % or s1 = e2*e3;
• s2 = max(e5,e6); % or s2= e5+e6>0;
• s3 = min(e4,s2); % or s3 = e4*s2;
• s4 = max(s1,s3); % or s4 = s1+s3 > 0;
• s5= min([e1;s4;e7]); % or s5=e1*s4*e7;
• count = count + s5;
• End;
• P_circuit_works = count/Stats

% our calculation: P(circuit_works)= 0.9.*(1-(1-0.5.*0.3).*(1-0.1.*(1-0.6.*0.5))).*0.8==0.15084
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<tr>
<td>Probability of component working</td>
<td>0.3</td>
<td>0.8</td>
<td>0.2</td>
<td>0.2</td>
<td>0.5</td>
<td>0.6</td>
<td>0.4</td>
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</table>
Prob. it works $= (1 - 0.4)^0 	imes 0.3 \times \left[ 1 - (1 - 0.8 \times 0.2) \times (1 - 0.2 \times 0.05) \right] \times 0.6 = 0.0269$
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<td>0.3</td>
<td>0.8</td>
<td>0.2</td>
<td>0.2</td>
<td>0.5</td>
<td>0.6</td>
<td>0.4</td>
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Prob circuit works =
= 0.4 \cdot 0.3 \cdot \left[ 1 - (1 - 0.8) \cdot (1 - 0.2) \right] \cdot \left[ 1 - (1 - 0.2) \cdot (1 - 0.5) \right] \cdot 0.6 = 0.363
Final probability: \(0.0269 + 0.0363 = 0.0632\)
QUESTIONS
Found in Google Autocomplete

- Why are there ants in my laptop?
- Why are there so many crows in Rochester, MN?
- Why are dogs afraid of fireworks?
- Why aren't there guns in Harry Potter?
- Why are there spiders in my house?
- Why is there an owl in my backyard?
- Why is there an owl outside my window?
- Why is there an owl on the dollar bill?
- Why do owls attack people?
- Why are AK-47s so expensive?
- Why are there helicopters circling my house?
- Why are there gods?
- Why are there two spoons?
- Why are cigarettes legal?
- Why are there ducks in my pool?
- Why is Jesus white?
- Why is there liquid in my ear?
- Why do tips feel good?
- Why do good people die?
- Why is there no GPS in laptops?
- Why is there no sound on my phone?
- Why do AR-15s sound good?
- Why do trees die?
- Why is programming so hard?
- Why is there a 60% chance of rain?
- Why are potatoes good?
- Why do trees like me?
- Why do bows like me?
- Why do my parents have to see this?
- Why do monkeys sound good?
- Why are whales jumping? Why are witches green?