Linear Programming (LP) with inequality constraints define a feasible region.

\[
\begin{align*}
\text{max } & 2x_1 + 3x_2 \\
\text{subject to } & x_1 + 2x_2 \leq 8 \\
& 3x_1 + 2x_2 \leq 12 \\
& x_1, x_2 \geq 0
\end{align*}
\]
The LP objective function is a hyperplane; optimization moves the plane away (max) from zero.

\[
\text{max } 2x_1 + 3x_2
\]

subject to

\[
x_1 + 2x_2 \leq 8
\]
\[
3x_1 + 2x_2 \leq 12
\]
\[
x_1, x_2 \geq 0
\]
Manual Solution by Enumeration

We know the optimal solution lies at a corner point. In 2D, we can enumerate all corner points.

\[
\begin{align*}
  f(0,0) &= 2(0) + 3(0) = 0 \\
  f(4,0) &= 2(4) + 3(0) = 8 \\
  f(0,4) &= 2(0) + 3(4) = 12 \\
  f(2,3) &= 2(2) + 3(3) = 13 \\
\end{align*}
\]

Let's say we start at (0,0). So long as we always move to an adjacent corner point with a greater objective value, we will end up at the optimal solution.

\[
(0,0) \rightarrow (0,4) \rightarrow (2,3)
\]

We can skip point (4,0).
The Simplex Method

Invented by Dantzig in 1947 (or earlier, the US Air Force prevented publication during WWII).

Independently invented by Kantorovich in 1939, but this was unknown outside of the Soviet Union.

Beginning with a feasible corner point, the Simplex Method identifies on those corner points that are
- adjacent to the current corner point
- have a better objective value

The cycle continues until no adjacent corner points improve the objective. This is a local optimum, which is global for an LP.
Step 0: Transform the problem into standard form.

\[
\begin{align*}
\text{max } & \quad 2x_1 + 3x_2 \\
\text{subject to} & \quad x_1 + 2x_2 \leq 8 \\
& \quad 3x_1 + 2x_2 \leq 12 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{max } f & = c^T x \\
\text{subject to} & \quad A x = b \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{max } & \quad 2x_1 + 3x_2 \\
\text{subject to} & \quad x_1 + 2x_2 + s_1 = 8 \\
& \quad 3x_1 + 2x_2 + s_2 = 12 \\
& \quad x_1, x_2, s_1, s_2 \geq 0
\end{align*}
\]

\[
x = \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix}, \quad c^T = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 8 \\ 12 \end{pmatrix}
\]
Step 1: Set up Initial Tableau

max \ 2x_1 + 3x_2

subject to
\ x_1 + 2x_2 + s_1 = 8 \\
3x_1 + 2x_2 + s_2 = 12 \\
x_1, x_2, s_1, s_2 \geq 0

\[ \begin{align*}
    x &= \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} \\
    c^T &= \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \\
    A &= \begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{pmatrix} \\
    b &= \begin{pmatrix} 8 \\ 12 \end{pmatrix}
\end{align*} \]

\[
\begin{pmatrix}
    A \\
    -c^T \\
    f \\
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2 \\
    s_1 \\
    s_2 \\
    f \\
\end{pmatrix}
\]

\[
\begin{array}{cccccc|c}
    x_1 & x_2 & s_1 & s_2 & f & \text{basic (nonzero) variables} \\
    1 & 2 & 1 & 0 & 0 & 8 \\
    3 & 2 & 0 & 1 & 0 & 12 \\
    -2 & -3 & 0 & 0 & 1 & 0 \\
\end{array}
\]

objective value
Step 2: While any variables have negative coefficients in the last row, choose the variable with the most negative.

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$s_1$</td>
<td>$s_2$</td>
<td>$f$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>-2</td>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Why? Variables with nonzeros in the last row are not in the basic solution; they are zero in our current solution. The last row is our objective function:

$$f = 2x_1 + 3x_2$$

To increase $f$, we want to make a zero variable nonzero by bringing it into the basic solution. We choose the variable with the fastest change in $f$, i.e. the most negative in the tableau.
Step 3: Bring the entering variable into the basic solution by zeroing out all rows except one. Which one? The row with the smallest ratio of the RHS to the coefficient of the entering variable.

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>-2</td>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

$8 / 2 = 4 \quad \leftarrow \text{choose this row}$

$12 / 2 = 6$

\[\begin{array}{cccccc}
1/2 & 1 & 1/2 & 0 & 0 & 4 \\
3 & 2 & 0 & 1 & 0 & 12 \\
-2 & -3 & 0 & 0 & 1 & 0 \\
\end{array}\]

\[\begin{array}{cccccc}
\frac{1}{2}R_1 \\
\frac{1}{2}
\end{array}\]

\[\begin{array}{cccccc}
1/2 & 1 & 1/2 & 0 & 0 & 4 \\
3 & 2 & 0 & 1 & 0 & 12 \\
-2 & -3 & 0 & 0 & 1 & 0 \\
\end{array}\]

\[\begin{array}{cccccc}
R_2 - 2R_1 \\
\end{array}\]

\[\begin{array}{cccccc}
1/2 & 1 & 1/2 & 0 & 0 & 4 \\
2 & 0 & -1 & 1 & 0 & 4 \\
-2 & -3 & 0 & 0 & 1 & 0 \\
\end{array}\]
Step 2: While any variables have negative coefficients in the last row, choose the variable with the most negative.

Step 3: Bring the entering variable into the basic solution by zeroing out all rows except one. Which one? The row with the smallest ratio of the RHS to the coefficient of the entering variable.

\[
\begin{array}{cccccc}
& x_1 & x_2 & s_1 & s_2 & f \\
1/2 & 1 & 1/2 & 0 & 0 & 4 & 4 / 0.5 = 8 \\
2 & 0 & -1 & 1 & 0 & 4 & 4 / 2 = 2 \rightarrow \text{choose this row} \\
-1/2 & 0 & 3/2 & 0 & 1 & 12 \\
\end{array}
\]

\[
\begin{array}{cccccc}
x_1 & x_2 & s_1 & s_2 & f \\
1/2 & 1 & 1/2 & 0 & 0 & 4 \\
1 & 0 & -1/2 & 1/2 & 0 & 2 \\
-1/2 & 0 & 3/2 & 0 & 1 & 12 \\
\end{array}
\]

\[
\begin{array}{cccccc}
x_1 & x_2 & s_1 & s_2 & f \\
0 & 1 & 3/4 & -1/4 & 0 & 3 \\
1 & 0 & -1/2 & 1/2 & 0 & 2 \\
0 & 0 & 5/4 & 1/4 & 1 & 13 \\
\end{array}
\]

entering variable
basic (nonzero) variables
objective value
Step 2: While any variables have negative coefficients in the last row, choose the variable with the most negative.

Since there are no more negative coefficients in the last row, we have found the optimal solution! Any change would not improve the objective.

The final solution can be read from the tableau:

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>3/4</td>
<td>-1/4</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>-1/2</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>5/4</td>
<td>1/4</td>
<td>1</td>
</tr>
</tbody>
</table>

$x_1 = 2$
$x_2 = 3$
$s_1, s_2 = 0$ (not in the basic solution)
$f = 13$