AE 353
EXAM # 1
Monday, October 16, 2017

DO NOT START UNTIL INSTRUCTED BY EXAMINER.

Exam Policy:

- open book and class-notes
- calculators allowed
- no cell-phones

Some Remarks:

- There are 3 problems. Read all questions and point values.
- Write your name on every sheet of paper you hand in.
- Max score is 100.
- Best of luck
Problem 1 (40pts)

Consider the inverted pendulum, torsional spring and damper system shown. At the position $\theta = 0$, the spring is relaxed.

1. Write the equations of motion.

2. For small perturbations around the equilibrium, find the transfer function, $H(s)$, from the input torque $T$ to the mass position $\theta$ as a function of $k$, $b$, $m$, $l$ and gravity $g$.

3. What is the minimum value of the stiffness $k$ as a function of the problem data to yield a stable system?

4. What is the transfer function, $G(s)$ from the input torque $T$ to the angular acceleration $\ddot{\theta}$?

\[ m l^2 \ddot{\theta} = T - k \theta - b \dot{\theta} + m gl \sin \theta \]

\[ \sin \theta \approx \theta \Rightarrow ml^2 \ddot{\theta} = T - k \theta - b \dot{\theta} + mgl \theta \]

\[ \Rightarrow \frac{\Theta(s)}{\dot{\Theta}(s)} = \frac{1}{ml^2 s^2 + bs + k - mgl} = H(s) \]

\[ k - mgl > 0 \quad \text{for stability} \quad \Rightarrow \quad k > mgl \]

\[ \frac{\ddot{\Theta}(s)}{\dot{T}(s)} = \frac{s^2 \dot{\Theta}(s)}{\dot{T}(s)} = s^2 H(s) \]

\[ = \frac{s^2}{ml^2 s^2 + bs + k - mgl} \]
Problem 2 (20pts)

Consider a standard 2nd order system $H(s)$. When the input is a unit step $1(t)$ the overshoot is 20% and the settling time 5 secs. Find $H(s)$.

$$H(s) = \frac{\omega_n^2}{s^2 + 2\xi \omega_n s + \omega_n^2}$$

$M_p = 20\% \Rightarrow \xi \approx 0.46$

from tables or from $M_p = e^{-\pi \xi / \sqrt{1-\xi^2}}$

$\xi = 0.46 \Rightarrow 3\omega_n = \frac{5}{0.46} \Rightarrow$

$\omega_n = \frac{5}{0.46} \Rightarrow 3 \approx 2$

$\therefore H(s) = \frac{4}{s^2 + 1.8s + 4}$
Problem 3 (40pts)

For the signal $u(t)$ sketched below, find its Laplace transform when $a = 1$, $b = 2$, $c = 2$, $d = 1$.

(i) If $u$ is an input to a stable

$$H(s) = \frac{s^{n-2} + b_n s^{n-3} + \ldots b_1 s + 1}{s^n + a_{n-1} s^{n-1} + \ldots a_1 s + 1}$$

what is the initial slope at $t = 0^+$ and what is steady state value of the output response to that input?

(ii) If $a = b = c = \frac{2}{\epsilon}$, $d = 0$, what are your answers to (i) as $\epsilon \to 0$?

$$u(t) = 2 \cdot 1(t) - r(t-1) + r(t-2) \Rightarrow$$

$$\hat{u}(s) = \frac{2}{s} - \frac{1}{s^2} + \frac{1}{s^2}$$

(i) \[ y(0^+) = s \frac{\hat{y}(s)}{s} \bigg|_{s \to 0} = s^2 H(s) \hat{u}(s) \bigg|_{s \to 0} = s^2 H(s) \frac{2}{s} \bigg|_{s \to 0} = 0 \]

\[ y(\infty) = s \frac{\hat{y}(s)}{s} \bigg|_{s \to 0} = s H(s) \hat{u}(s) \bigg|_{s \to 0} = 1 \]

(ii) \[ \text{As } \epsilon \to 0 \quad u(t) \to 2 \delta(t) \Rightarrow \]

\[ y(0^+) = s^2 H(s) \frac{2}{s} \bigg|_{s \to 0} = 2 \]

\[ y(\infty) = s H(s) \frac{2}{s} \bigg|_{s \to 0} = 0 \]