SOL

AE 353
EXAM # 1
Monday, October 16, 2017

DO NOT START UNTIL INSTRUCTED BY EXAMINER

Exam Policy:

- open book and class-notes
- calculators allowed
- no cell-phones

Some Remarks:

- There are 3 problems. Read all questions and point values.
- Write your name on every sheet of paper you hand in.
- Max score is 100.
- Best of luck

Problem 1 (40pts)

Consider the inverted pendulum, torsional spring and damper system shown. At the position $\theta = 0$, the spring is relaxed.

1. Write the equations of motion.
2. For small perturbations around the equilibrium, find the transfer function, $H(s)$, from the input torque T to the mass position θ as a function of k, b, m, l and gravity g.
3. What is the minimum value of the stiffness k as a function of the problem data to yield a stable system?
4. what is the transfer function, $G(s)$ from the input torque T to the angular acceleration $\ddot{\theta}$

1. $ml^2\ddot{\theta} = T - k\theta - b\dot{\theta} + mgl\sin\theta$

2. $\sin\theta \approx \theta \Rightarrow ml^2\ddot{\theta} = T - k\theta - b\dot{\theta} + mgl\theta \Rightarrow$

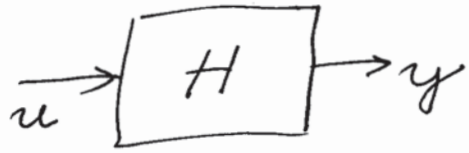
$$\Rightarrow \frac{\hat{\theta}(s)}{\hat{T}(s)} = \frac{1}{ml^2s^2 + bs + k - mgl} = H(s)$$

3. $k - mgl > 0$ for stability $\Rightarrow$

$k > mgl$

4. $$\frac{\hat{\ddot{\theta}}(s)}{\hat{T}(s)} = \frac{s^2\,\hat{\theta}(s)}{\hat{T}(s)} = s^2 H(s)$$

$$= \frac{s^2}{ml^2s^2 + bs + k - mgl}$$



Problem 2 (20pts)

Consider a *standard* 2nd order system $H(s)$. When the input is a unit step $1(t)$ the overshoot is 20% and the settling time 5 secs. Find $H(s)$.

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

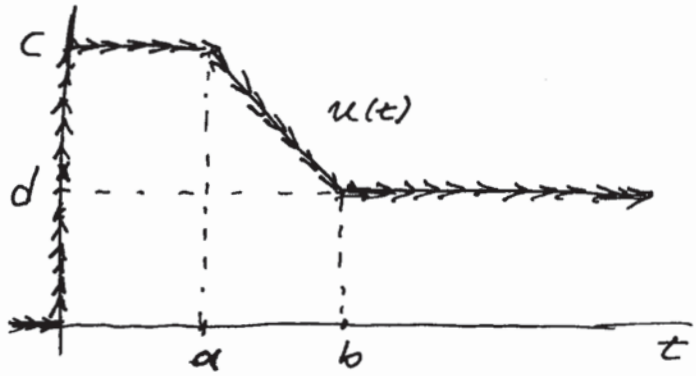
$$M_p = 20\% \Rightarrow \zeta \simeq 0.46$$

from tables or from $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$

$$t_s = 5 = \frac{4.6}{\zeta\omega_n} \Rightarrow \zeta\omega_n = \frac{5}{4.6} \Rightarrow$$

$$\omega_n = \frac{5}{4.6}\zeta \approx 2$$

$$\therefore \quad H(s) = \frac{4}{s^2 + 1.8s + 4}$$



Problem 3 (40pts)

For the signal $u(t)$ sketched below, find its Laplace transform when $a = 1$, $b = 2$, $c = 2$, $d = 1$.

(i) If u is an input to a stable

$$H(s) = \frac{s^{n-2} + b_{n-3}s^{n-3} + \ldots b_1 s + 1}{s^n + a_{n-1}s^{n-1} + \ldots a_1 s + 1}$$

what is the initial slope at $t = 0^+$ and what is steady state value of the output response to that input?

(ii) If $a = b = \epsilon$, $c = \frac{2}{\epsilon}$, $d = 0$, what are your answers to (i) as $\epsilon \to 0$?

$$u(t) = 2 \cdot 1(t) - r(t-1) + r(t-2) \Rightarrow$$

$$\hat{u}(s) = 2\frac{1}{s} - e^{-s}\frac{1}{s^2} + e^{-2s}\frac{1}{s^2}$$

(i) $$\dot{y}(0^+) = s\, s\hat{y}(s)\Big|_{s\to\infty} = s^2 H(s)\hat{u}(s)\Big|_{s\to\infty}$$

$$= s^2 H(s)\, 2\frac{1}{s}\Big|_{s\to\infty} = 0$$

$$y(\infty) = s\hat{y}(s)\Big|_{s\to 0} = s H(s)\hat{u}(s)\Big|_{s\to 0} = 1$$

(ii) As $\varepsilon \to 0$ $u(t) \to 2\delta(t) \Rightarrow$

$$\dot{y}(0^+) = s^2 H(s)\, 2\cdot 1\Big|_{s\to\infty} = 2$$

$$y(\infty) = s H(s)\, 2\cdot 1\Big|_{s\to 0} = 0$$