University of Illinois **Spring 2013**

ECE 586BH: Problem Set 5

Multistage games, including repeated games, with observed moves

Due: Thursday, April 11 at beginning of class

Reading: Fudendberg and Triole, Sections 3.2 and 5.1. See Chapter 4 for examples.
Ozdaglar lecture notes 12,13,15, & 16 (on extensive form games and repeated games)

1. **[Another subgame perfect equilibria in repeated prisoner's dilemma]**
Consider the repeated game with the stage game being prisoner's dilemma with the payoff functions g_i given by

	c	d
c	1,1	-1,2
d	2,-1	0,0

. Action c means to cooperate. Suppose player 1 has discount factor δ_1 and player 2 has discount factor δ_2, so the payoff for player i is given by $J_i(a) = \sum_{t=0}^{\infty} \delta_i^t g_i(a(t))$, for $a = ((a_1(t), a_2(t)) : t \geq 0)$ with $a_i(t) \in \{c, d\}$. Consider the following pair of scripts, for play beginning at time $t = 0$:

player 1: $c, d, d, c, c, c, c, \ldots$
player 2: $d, d, d, d, c, c, c, \ldots$

Both scripts have players playing c for $t \geq 4$. Let μ_i^* be the trigger strategy that follows the script for player i as long as both players have followed the script in the past. If either player has deviated from the script in the past, player i always plays d. Let $\mu^* = (\mu_1^*, \mu_2^*)$.

   (a) Express $J_i(\mu^*)$ in terms of δ_i for $i \in \{1, 2\}$.

   (b) For what values of (δ_1, δ_2) is μ^* a subgame perfect equilibrium (SPE) of the repeated game? Justify your answer using the one-stage-deviation principle.

2. **[Feasible payoffs for a trigger strategy in two stage game for a stage game with multiple NEs]**
Consider a multistage game such that the stage game has the following payoff matrix, where player 1 is the row player and player 2 is the column player:

	1	2	3
1	4,4	5,3	9,3
2	3,5	6,6	9,2
3	3,9	2,9	8,8

   (a) Identify two NE for the stage game.

   (b) Consider a two stage game obtained by playing the above stage game twice, with discount factor equal to one, so payoffs for the repeated game are equal to the sum of payoffs for the two stages. Describe a subgame perfect equilibrium for the two stage game so that the payoff vector is $(14, 14)$. Verify that the strategy profile is subgame perfect using the single-deviation principle.

3. **[Repeated play of Cournot game]**
Consider the repeated game with the stage game being two-player Cournot competition with payoffs $g_i(s) = s_i(a - s_1 - s_2 - c)$, where $a > c > 0$. Here s_i represents a quantity produced by firm i, the market price is $a - s_1 - s_2$ and c is the production cost per unit produced. The action space for each player is the interval $[0, \infty)$.

   (a) Identify the pure strategy NE of the stage game and the corresponding payoff vector.

(b) Identify the maxmin value $\underline{v}_i$ for each player for the stage game.

(c) Identify the feasible payoff vectors for the stage game.

(d) Identify the payoff vectors for the repeated game that are feasible for Nash equilibrium of the repeated game for the discount factor δ sufficiently close to one, guaranteed by Nash's general feasibility theorem.

(e) Identify the payoff vectors for the repeated game that are feasible for subgame perfect equilibrium of the repeated game for the discount factor δ sufficiently close to one, guaranteed by Friedman's general feasibility theorem.

(f) Identify the payoff vectors for the repeated game that are feasible for subgame perfect equilibrium of the repeated game for the discount factor δ sufficiently close to one, guaranteed by the Fudenberg/Maskin general feasibility theorem.

4. **[A Cournot game with incomplete information]**
Given $c > a > 0$ and $0 < p < 1$, consider the following version of a Cournot game. Suppose the type θ_1 of player 1 is either zero, in which case his production cost is zero, or one, in which case his production cost is c (per unit produced). Player 2 has only one possible type, and has production cost is c. Player 1 knows his type θ_1. It is common knowledge that player 2 believes player 1 is type one with probability p. Both players sell what they produce at price per unit $a - q_1 - q_2$, where q_i is the amount produced by player i. A strategy for player 1 has the form $(q_{1,0}, q_{1,1})$, where q_{1,θ_1} is the amount produced by player one if player 1 is type θ_1. A strategy for player 2 is q_2, the amount produced by player 2.

(a) Identify the best responses of each player to strategies of the other player.

(b) Identify the Bayes-Nash equilibrium of the game. *For simplicity, assume* $a \geq 2c$. (Note: After finding the equilibrium, it can be shown that the three corresponding payoffs are given by

$$u_2(q^{NE}) = \frac{(a - (2-p)c)^2}{9}$$

and

$$u_1(q^{NE}|\theta_1 = 0) = \frac{(2a + (2-p)c)^2}{36}, \quad u_1(q^{NE}|\theta_1 = 1) = \frac{(2a - (1+p)c)^2}{36}.$$

As expected, $u_2(q^{NE})$ is increasing in p, and both $u_1(q^{NE}|\theta_1 = 0)$ and $u_1(q^{NE}|\theta_1 = 1)$ are decreasing in p with $u_2(q^{NE}) < u_1(q^{NE}|\theta_1 = 1) < u_1(q^{NE}|\theta_1 = 0)$ for $0 < p < 1$. In the limit $p = 1$, $u_2(q^{NE}) = u_1(q^{NE}|\theta_1 = 1) = \frac{(a-c)^2}{9}$, as in the original Cournot game with complete information and production cost c for both players.)