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ECE 552, Fall 2017
HW # 9 Solutions

Prob. 9.1: $x_n = \frac{4}{3} x_{n-1} - \frac{1}{3} x_{n-2} + \frac{2}{3} h \dot{x}_n$

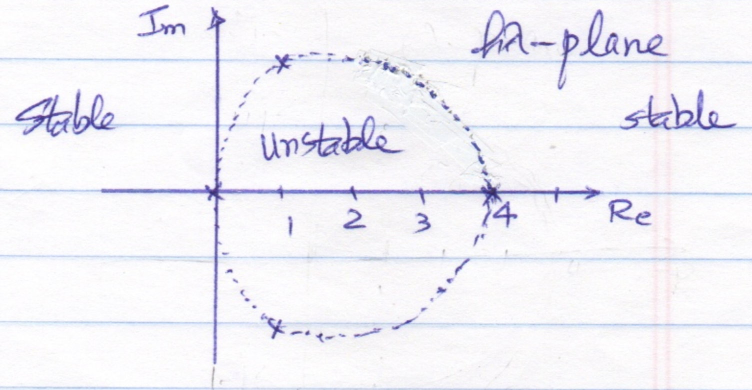
(a) $-x_n + \frac{4}{3} x_{n-1} - \frac{1}{3} x_{n-2} + \frac{2}{3} h\lambda x_n$

$$P(z, h\lambda) = -z^2 + \frac{4}{3} z - \frac{1}{3} + \frac{2}{3} \lambda z^2$$

$$h\lambda = \frac{z^2 - \frac{4}{3} z + \frac{1}{3}}{\frac{2}{3} z^2}$$

(b)

z	$h\lambda$
1	0
-1	4
j	1+j2
0	∞



(c) Suitable for stiff systems. Stable to the left of a vertical line in the $h\lambda$-plane (the imaginary axis) and damped at ∞.

Prob. 9.2 $x_n = -4x_{n-1} + 5x_{n-2} + h(-2\dot{x}_n + 8\dot{x}_{n-1})$

(a) $k = 2$ (has x_{n-2})

(b) Order 2 (See HW#8)

(c) Implicit (includes $\dot{x}_n$)

(d) Consistent : Order > 1

(e) Zero-stable? $-z^2 - 4z + 5 = 0$, $z = 1, -5$

Not zero-stable

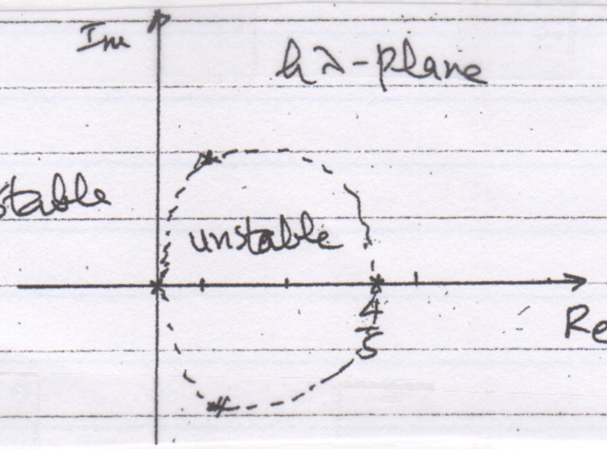
(f) Not convergent (because it is not zero-stable)

(g) $-x_n - 4x_{n-1} + 5x_{n-2} + h\lambda(-2x_n + 8x_{n-1})$

$$P(z, h\lambda) = -z^2 - 4z + 5 + h\lambda(-2z^2 + 8z) = 0$$

(h) $h\lambda = \dfrac{z^2 + 4z - 5}{-2z^2 + 8z}$

z	$h\lambda$
1	0
-1	$4/5$
j	$\frac{5+j14}{17}$
0	∞

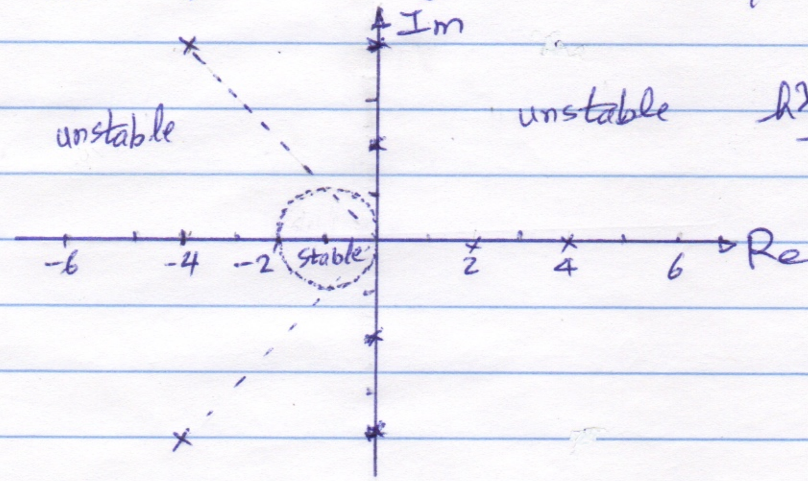


(i) Not suitable for stiff systems : Not convergent and not L-stable - includes $\dot{x}_{n-1}$

In fact the formula is not suitable for transient analysis because it is not zero-stable

Prob. 9.3

(1) F.E. Region of Stability in the $h\lambda$-plane:



(a) $\lambda_1 = -2, \lambda_2 = -4, \lambda_3 = -4+j4, \lambda_4 = -4-j4$ (Absolutely stable)

$0 < h < \frac{1}{4}$ (To bring all λs inside the stable region in the $h\lambda$-plane)

(b) $\lambda_1 = +2, \lambda_2 = +4, \lambda_3 = +j4, \lambda_4 = -j4$ (unstable)

$h > 0$; $h\lambda$s in unstable region

(c) $\lambda_1 = -2, \lambda_2 = -4, \lambda_3 = +j4, \lambda_4 = -j4$ (oscillatory)

No $h > 0$ can bring $+j4$ & $-j4$ to boundary in $h\lambda$-plane.

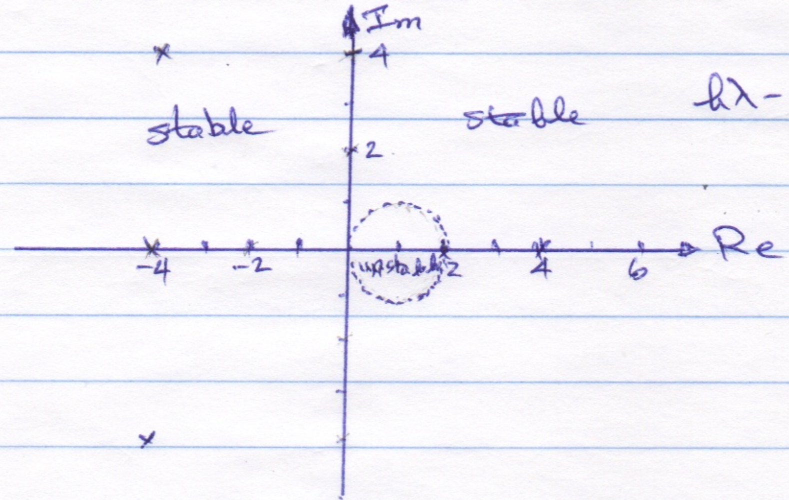
(d) $\lambda_1 = -2j, \lambda_2 = -2j, \lambda_3 = +2j, \lambda_4 = +2j$

(unstable, double root on $j\omega$-axis)

$h > 0$; $h\lambda$s stay in unstable region

Prob. 9.3 (cont.)

(2) B.E. Region of stability in $h\lambda$-plane:



(a) $\lambda_1 = -2,\ \lambda_2 = -4,\ \lambda_3 = -4+j4,\ \lambda_4 = -4-j4$ (Absolutely stable)

$h > 0$ All $h\lambda_s$ stay in stable region

(b) $\lambda_1 = +2,\ \lambda_2 = +4,\ \lambda_3 = +j4,\ \lambda_4 = -j4$ (unstable)

$h < 1$: Need to bring 'closest' λ in rhp inside the unstable region. You only need to bring one λ inside the unstable region.

(c) $\lambda_1 = -2,\ \lambda_2 = -4,\ \lambda_3 = +j4,\ \lambda_4 = -j4$ (oscillatory)

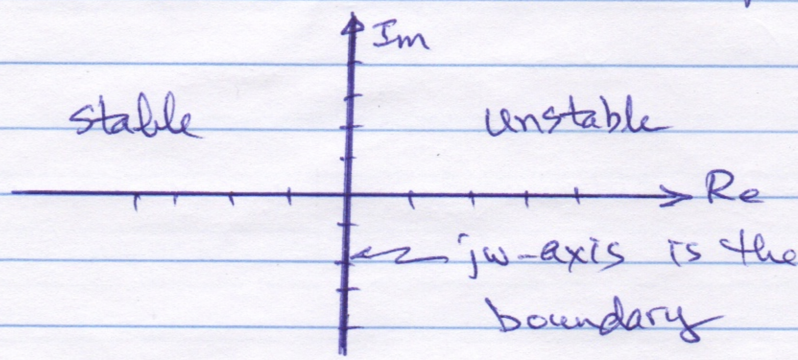
No $h > 0$ can bring $+j4$ & $-j4$ to boundary in $h\lambda$-plane.

(d) $\lambda_1 = -2j,\ \lambda_2 = -2j,\ \lambda_3 = +2j,\ \lambda_4 = +2j$ (unstable)

No $h > 0$ can bring double root on $j\omega$-axis to inside the unstable region in $h\lambda$-plane

Prob. 9.3 (Cont.)

(3) T.R. Region of stability in $h\lambda$-plane:



$h > 0$ in all cases (a), (b), (c) & (d)

Prob. 9.4

$$q_1 = v_1^3 + v_2$$
$$q_2 = v_1 + v_2^3$$

(a) B.E. $\left(\dot{x} = \frac{1}{h}x_n - \frac{1}{h}x_{n-1}\right)$ $\quad i = \frac{dq}{dt} \Rightarrow i_1 = \frac{1}{h}(q_n - q_{n-1})$

$$i_1 = \frac{1}{h}\left(v_{1,n}^3 + v_{2,n}\right) - \frac{1}{h}q_{1,n-1}, \quad q_{1,n-1} = \left(v_{1,n-1}^3 + v_{2,n-1}\right)$$

$$i_2 = \frac{1}{h}\left(v_{1,n} + v_{2,n}^3\right) - \frac{1}{h}q_{2,n-1}, \quad q_{2,n-1} = \left(v_{1,n-1} + v_{2,n-1}^3\right)$$

(b) $$i_1 = \frac{1}{h}\left[v_{1,n}^{3(k)} + 3v_{1,n}^{2(k)}\left(v_{1,n} - v_{1,n}^{(k)}\right) + v_{2,n}\right] - \frac{1}{h}q_{1,n-1}$$

$$= \frac{1}{h}\left[1 + 3(v_{1,n} - 1) + v_{2,n}\right] - \frac{1}{h}q_{1,n-1}$$

$$= \frac{1}{h}\left[3v_{1,n} + v_{2,n} - 2\right] - \frac{1}{h}q_{1,n-1}$$

9.4 (b) cont.

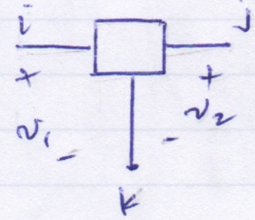
$$i_2 = \frac{1}{h}\left(q_{2,n} - q_{2,n-1}\right)$$

$$= \frac{1}{h}\left[v_{1,n} + v_{2,n}^{3(k)} + 3v_{2,n}^{2(k)}\left(v_{2,n} - v_{2,n}^{(k)}\right)\right] - \frac{1}{h}q_{2,n-1}$$

$$= \frac{1}{h}\left[v_{1,n} + 8 + 3\times 4\left(v_{2,n} - 2\right)\right] - \frac{1}{h}q_{2,n-1}$$

$$= \frac{1}{h}\left[v_{1,n} + 12v_{2,n} - 16\right] - \frac{1}{h}q_{2,n-1}$$

(c)



$$\begin{bmatrix} \frac{3}{h} & \frac{1}{h} & -\frac{4}{h} \\ \frac{1}{h} & \frac{12}{h} & -\frac{13}{h} \\ -\frac{4}{h} & -\frac{13}{h} & \frac{17}{h} \end{bmatrix} \begin{bmatrix} v_i \\ v_j \\ v_k \end{bmatrix} = \begin{bmatrix} \frac{2}{h} + \frac{q_{1,n-1}}{h} \\ \frac{16}{h} + \frac{q_{2,n-1}}{h} \\ -\frac{18}{h} - \frac{q_{1,n-1}}{h} - \frac{q_{2,n-1}}{h} \end{bmatrix}$$

(d) BDF, 2nd-order: $\dot{x}_n = \frac{3}{2h}\left(x_n - \frac{4}{3}x_{n-1} + \frac{1}{3}x_{n-2}\right)$

$$i_1 = \frac{3}{2h}\left(q_n - \frac{4}{3}q_{n-1} + \frac{1}{3}q_{1,n-2}\right)$$

$$= \frac{3}{2h}\left(v_{1,n}^3 + v_{2,n}\right) + \frac{1}{h}\left(-2q_{1,n-1} + \frac{1}{2}q_{1,n-2}\right)$$

$$i_2 = \frac{3}{2h}\left(v_{1,n} + v_{2,n}^3\right) + \frac{1}{h}\left(-2q_{2,n-1} + \frac{1}{2}q_{2,n-2}\right)$$

9.4 (d) cont.

Linearization:

$$i_1 = \frac{3}{2h}\left[3v_{1,n} + v_{2,n} - 2\right] + \frac{1}{h}\left(-2q_{,n-1} + \frac{1}{2}q_{,n-2}\right)$$

$$i_2 = \frac{3}{2h}\left[v_{1,n} + 12v_{2,n} - 16\right] + \frac{1}{h}\left(-2q_{2,n-1} + \frac{1}{2}q_{2,n-2}\right)$$

$$\begin{bmatrix} \frac{9}{2h} & \frac{3}{2h} & \frac{-12}{2h} \\ \frac{3}{2h} & \frac{18}{h} & \frac{-39}{2h} \\ \frac{-6}{h} & \frac{-39}{2h} & +\frac{51}{2h} \end{bmatrix} \begin{bmatrix} v_i \\ v_j \\ v_k \end{bmatrix} = \begin{bmatrix} \frac{3}{h} + \frac{2}{h}q_{1,n-1} - \frac{1}{2h}q_{1,n-2} \\ \frac{24}{h} + \frac{2}{h}q_{2,n-1} - \frac{1}{2h}q_{2,n-2} \\ -\frac{27}{h} - \frac{2}{h}q_{1,n-1} + \frac{1}{2h}q_{1,n-2} - \frac{2}{h}q_{2,n-2} + \frac{1}{2h}q_{2,n-2} \end{bmatrix}$$