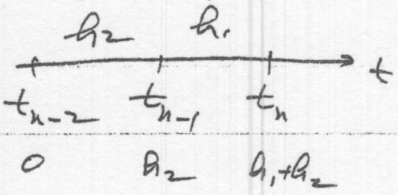
Prob. 8-1 (Solution)

$$x_n = \alpha_1 x_{n-1} + \alpha_2 x_{n-2} + h \beta_0 \dot{x}_n$$



(a) 1: $\quad 1 = \alpha_1 + \alpha_2 + 0 \quad ①$

t: $\quad t_n = \alpha_1 t_{n-1} + \alpha_2 t_{n-2} + h_1 \beta_0$

or, $\quad (h_1+h_2) = \alpha_1 h_2 + 0 + h_1 \beta_0 \quad ②$

t^2: $\quad t_n^2 = \alpha_1 t_{n-1}^2 + \alpha_2 t_{n-2}^2 + h_1 \beta_0 \, 2t_n$

or, $\quad (h_1+h_2)^2 = \alpha_1 h_2^2 + 0 + 2h_1\beta_0 (h_1+h_2) \quad ③$

From ② & ③: $\quad \beta_0 = \dfrac{h_1+h_2}{2h_1+h_2}$

$$\alpha_1 = \frac{(h_1+h_2)^2}{h_2(2h_1+h_2)}$$

From ①: $\quad \alpha_2 = \dfrac{-h_1^2}{-h_2(2h_1+h_2)}$

(b) If $h_1 = h_2 = h \Rightarrow \alpha_1 = \frac{4}{3}, \; \alpha_2 = -\frac{1}{3}, \; \beta_0 = \frac{2}{3}$

(c) LTE:

(i) Taylor Series Expansion:

$$x(t_{n-1}) = x(t_n) - h_1 \dot{x}(t_n) + \frac{h_1^2}{2}\ddot{x}(t_n) - \frac{h_1^3}{6}\dddot{x}(t_n) + \cdots$$

$$x(t_{n-2}) = x(t_n) - (h_1+h_2)\dot{x}(t_n) + \frac{(h_1+h_2)^2}{2}\ddot{x}(t_n) - \frac{(h_1+h_2)^3}{6}\dddot{x}(t_n) + \cdots$$

$$LTE = -x(t_n) + \alpha_1 x(t_{n-1}) + \alpha_2 x(t_{n-2}) + h\beta_0 \dot{x}(t_n)$$

Prob. 8.1 (cont.)

$$LTE = -x(t_n) + \frac{(h_1+h_2)^2}{h_2(2h_1+h_2)} x(t_{n-1}) - \frac{h_1^2}{h_2(2h_1+h_2)} x(t_{n-2}) + h_1 \frac{(h_1+h_2)}{2h_1+h_2} \dot{x}(t_n)$$

$$= -x(t_n) + \frac{(h_1+h_2)^2}{h_2(2h_1+h_2)}\Big[x(t_n) - h_1\dot{x}(t_n) + \frac{h_1^2}{2}\ddot{x}(t_n) - \frac{h_1^3}{6}\dddot{x}(t_n) + \cdots$$

$$- \frac{h_1^2}{h_2(2h_1+h_2)}\Big[x(t_n) - (h_1+h_2)\dot{x}(t_n) + \frac{(h_1+h_2)^2}{2}\ddot{x}(t_n) - \frac{(h_1+h_2)^3}{6}\dddot{x}(t_n) + \cdots$$

$$+ \frac{h_1(h_1+h_2)}{2h_1+h_2}\dot{x}(t_n)$$

$$= \frac{-h_1^2(h_1+h_2)^2}{6(2h_1+h_2)}\dddot{x}(t_n) + \cdots$$

OR, (ii) Put t^3 in formula :

$$-(h_1+h_2)^3 + \frac{(h_1+h_2)^2}{h_2(2h_1+h_2)} \cdot h_2^3 - \frac{h_1^2}{h_2(2h_1+h_2)} \times 0 + h_1\frac{(h_1+h_2)}{2h_1+h_2} \cdot 3(h_1$$

$$= \frac{-h_1^2(h_1+h_2)^2}{2h_1+h_2}$$

$$LTE = \frac{1}{3!}\left(\frac{-h_1^2(h_1+h_2)^2}{2h_1+h_2}\right)\dddot{x}(t_n)$$

If $h_1 = h_2 = h$,

$$LTE = -\frac{2}{9}h^3\dddot{x}(t_n)$$

Prob. 8.2

$$x_n = \alpha_1 x_{n-1} + \alpha_2 x_{n-2} + \alpha_3 x_{n-3} + h\beta_0 \dot{x}_n$$

(a) $k=3$; includes x_{n-3}.

(b) 1 : $\quad 1 = \alpha_1 + \alpha_2 + \alpha_3$. $\qquad$ (1)

t : $\quad t_n = \alpha_1 t_{n-1} + \alpha_2 t_{n-2} + \alpha_3 t_{n-3} + h\beta_0$

$$3h = \alpha_1(2h) + \alpha_2 h + 0 + h\beta_0$$

$$3 = 2\alpha_1 + \alpha_2 + \beta_0 \qquad (2)$$

t^2 : $\quad t_n^2 = \alpha_1 t_{n-1}^2 + \alpha_2 t_{n-2}^2 + \alpha_3 t_{n-3}^2 + h\beta_0 (2t_n)$

$$(3h)^2 = \alpha_1 (2h)^2 + \alpha_2 h^2 + 0 + h\beta_0 2(3h)$$

$$9 = 4\alpha_1 + \alpha_2 + 6\beta_0 \qquad (3)$$

t^3 : $\quad (3h)^3 = \alpha_1 (2h)^3 + \alpha_2 h^3 + 0 + h\beta_0 (3(3h)^2)$

$$27 = 8\alpha_1 + \alpha_2 + 0 + 27\beta_0 \qquad (4)$$

Solving (1), (2), (3) and (4) gives: $\alpha_1 = \frac{18}{11}$, $\alpha_2 = -\frac{9}{11}$, $\alpha_3 = \frac{2}{11}$, $\beta_0 = \frac{6}{11}$

$$x_n = \frac{1}{11}\left(18x_{n-1} - 9x_{n-2} + 2x_{n-3} + 6h\dot{x}_n\right)$$

(c) Implicit: Includes $\dot{x}_n$

(d) t^4: $\quad (3h)^4 = \alpha_1 (2h)^4 + \alpha_2 h^4 + 0 + h\beta_0 4(3h)^3$

$$C_{p+1} = -81 + \frac{16}{11}\times 18 - \frac{9}{11} + \frac{6}{11}\times 108 = \frac{36}{11}$$

$$\text{LTE} = \frac{1}{(p+1)!} C_{p+1} h^{p+1} \frac{d^{p+1}x}{dt^{p+1}} = \frac{1}{4!} \times \frac{36}{11} h^4 \frac{d^4x}{dt^4} = \frac{3}{22} h^4 \frac{d^4x}{dt^4}$$

Prob. 8.3

$$x_n = -4x_{n-1} + 5x_{n-2} + h(-2\dot{x}_n + 8\dot{x}_{n-1})$$

$k = 2$

(a) Order:

$1: \quad -1 - 4 + 5 = 0$ ✓

$t: \quad -t_n - 4t_{n-1} + 5t_{n-2} + h(-2+8)$

$\quad -2h - 4h + 0 + 6h = 0$ ✓

$t^2: \quad -t_n^2 - 4t_{n-1}^2 + 5t_{n-2}^2 + h(-4t_n + 16t_{n-1})$

$\quad -4h^2 - 4h^2 + 0 + h(-8h + 16h) = 0$ ✓

$t^3: \quad -8h^3 - 4h^3 - 0 + h(-24h^2 + 24h^2) =$

$-12h^3 \neq 0$

Order = 2

(b) Implicit: Includes $\dot{x}_n$

(c) $\text{LTE} = \frac{1}{3!}(-12h^3)\frac{d^3x}{dt^3} = -2h^3\frac{d^3x}{dt^3}$