ECE 552
Numerical Circuit Analysis

Chapter Six

NONLINEAR DC ANALYSIS

OR: Solution of Nonlinear Algebraic Equations

I. Hajj 2017

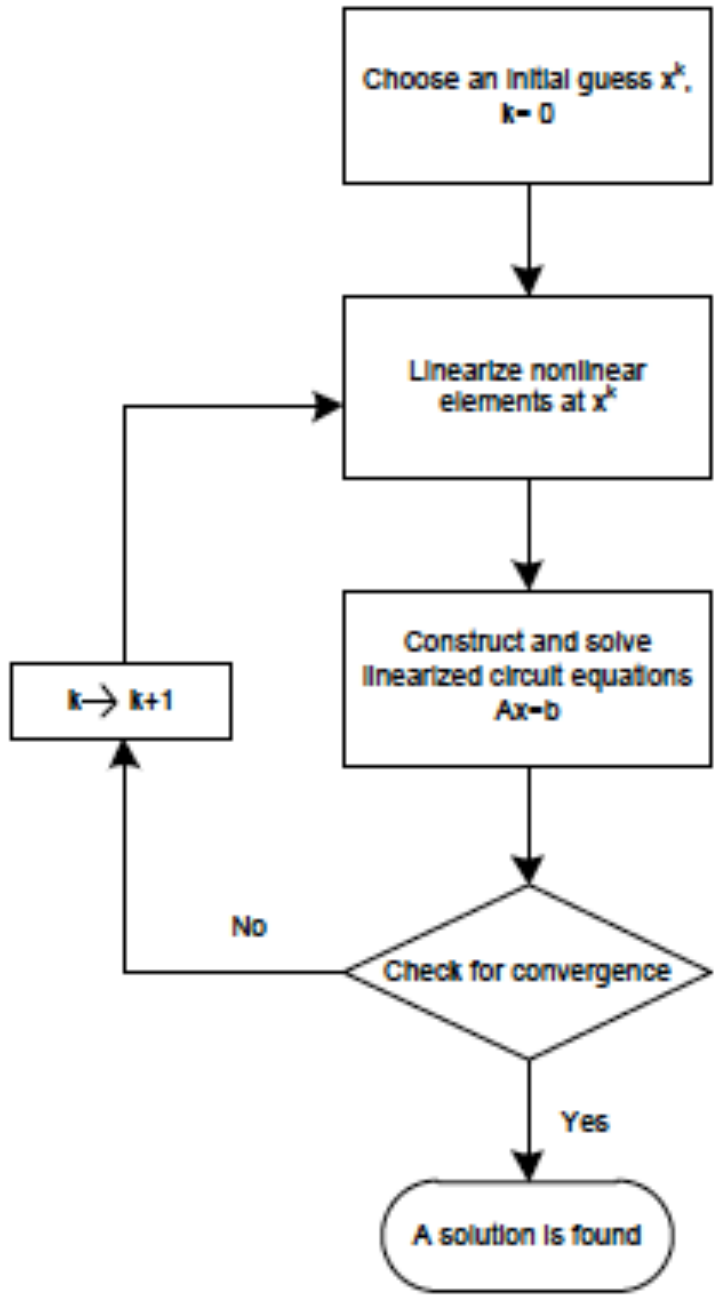


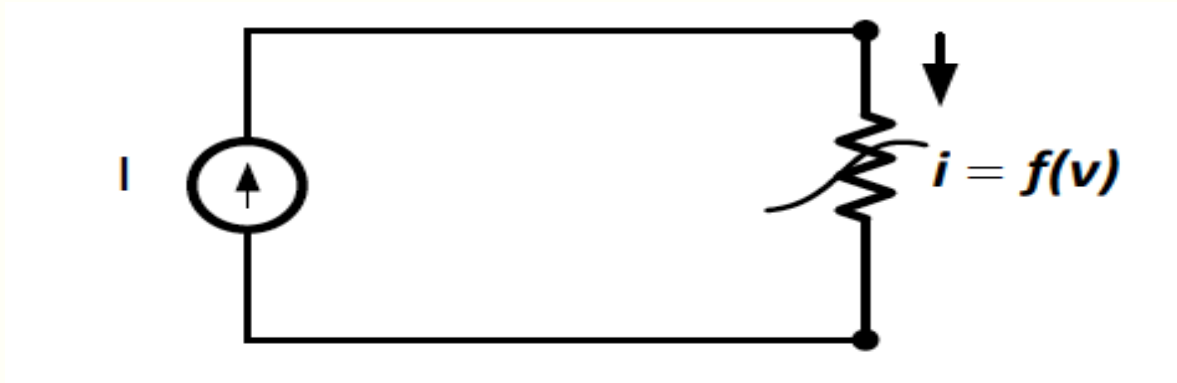
Figure 6.3.1: A flowchart of DC solution algorithm.

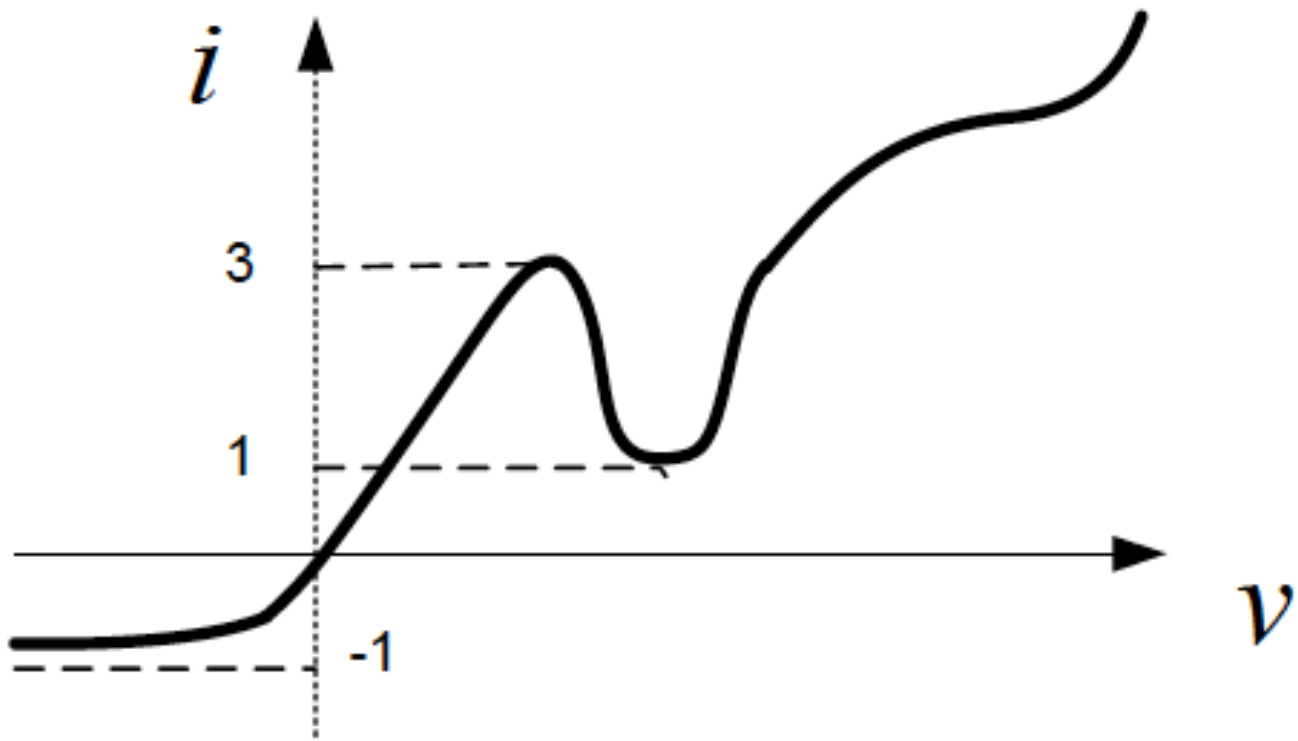
Nonlinear Algebraic Equations

- A system of linear equations **Ax = b** has a unique solution, unless **A** is singular.

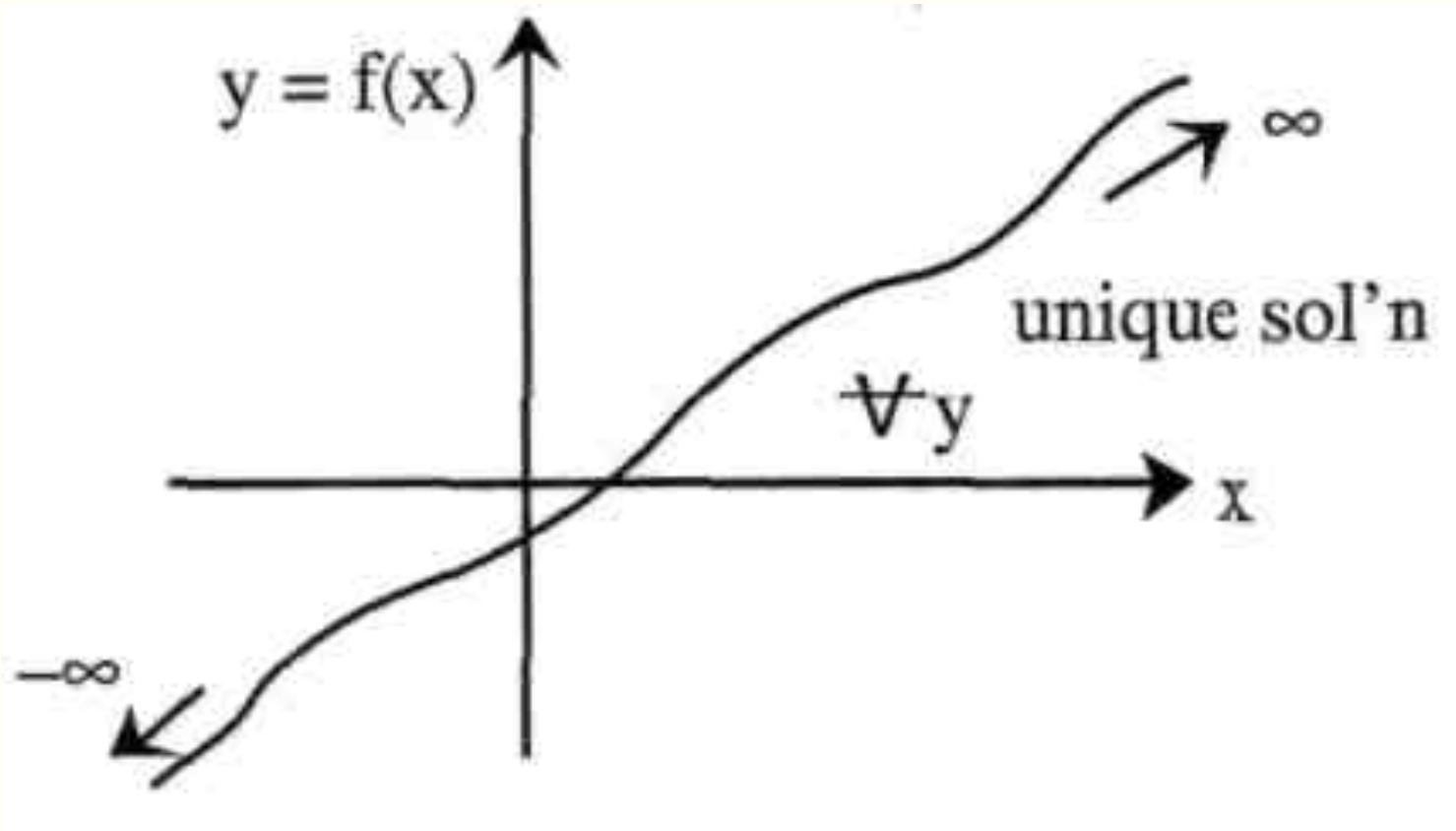
- However, a system of nonlinear equations **f(x) =y** may have *one solution, multiple finite solutions, no solution,* or *infinite number of solutions.*

Example

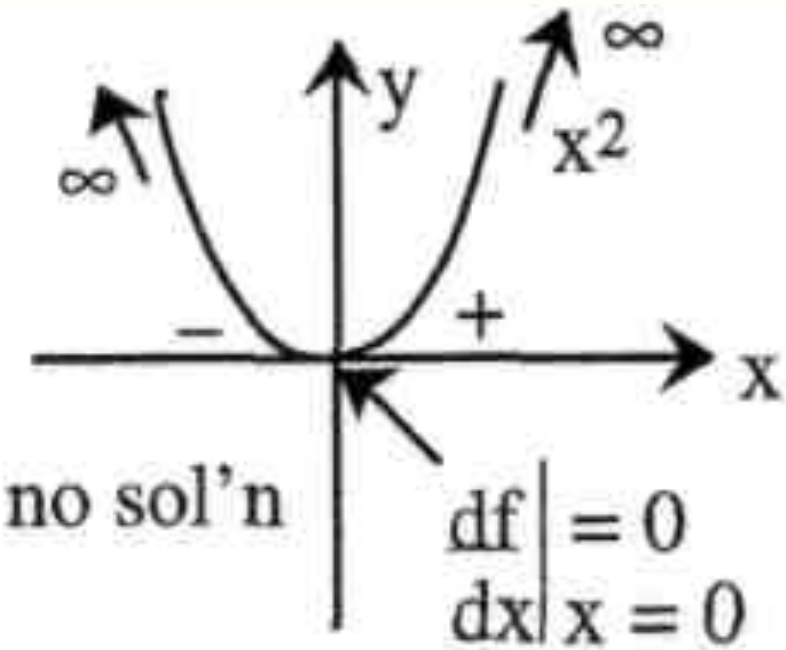




Example

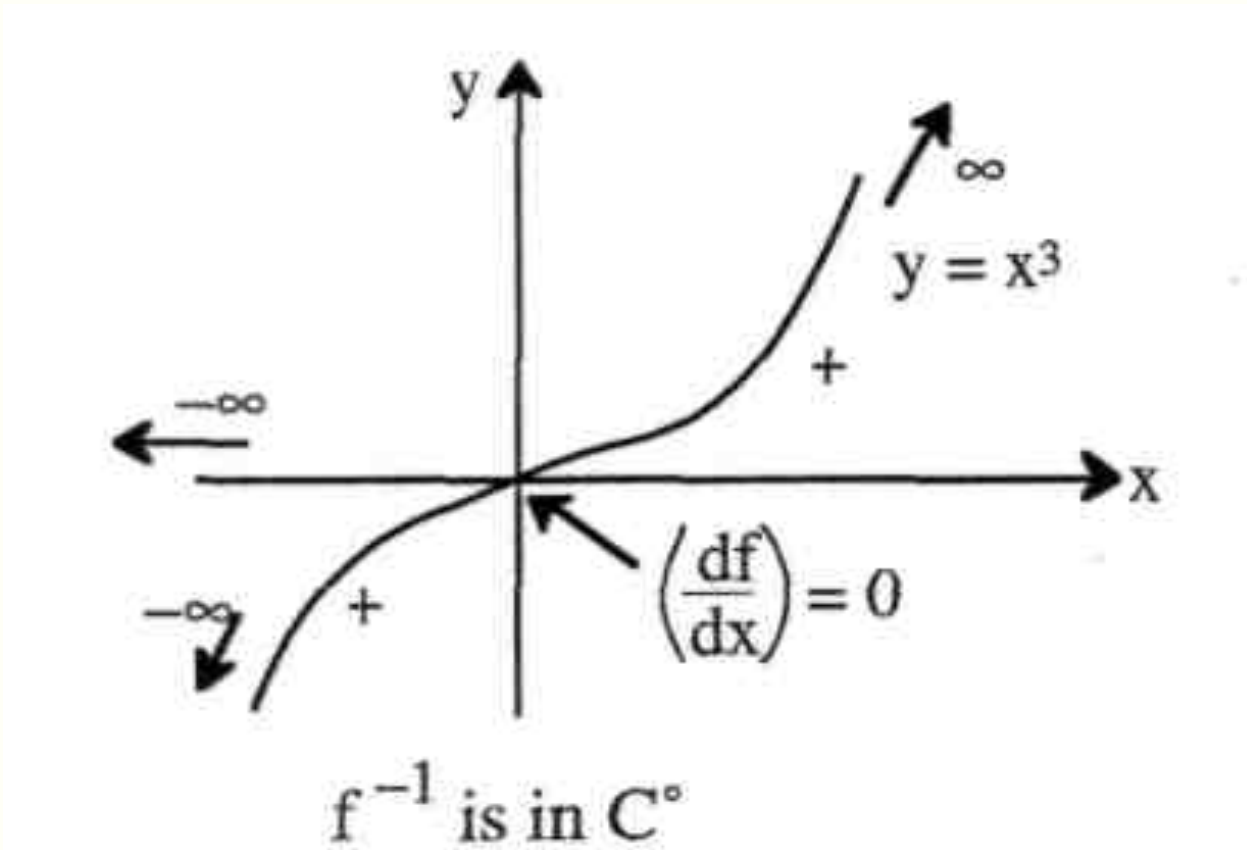


Example



$y = x^2$ has two solutions for $y > 0$
one solution for $y = 0$
no solution for $y < 0$

Example



$y = x^3$ has a unique solution for every y.

n-dimensional case

- **f(x) = y**

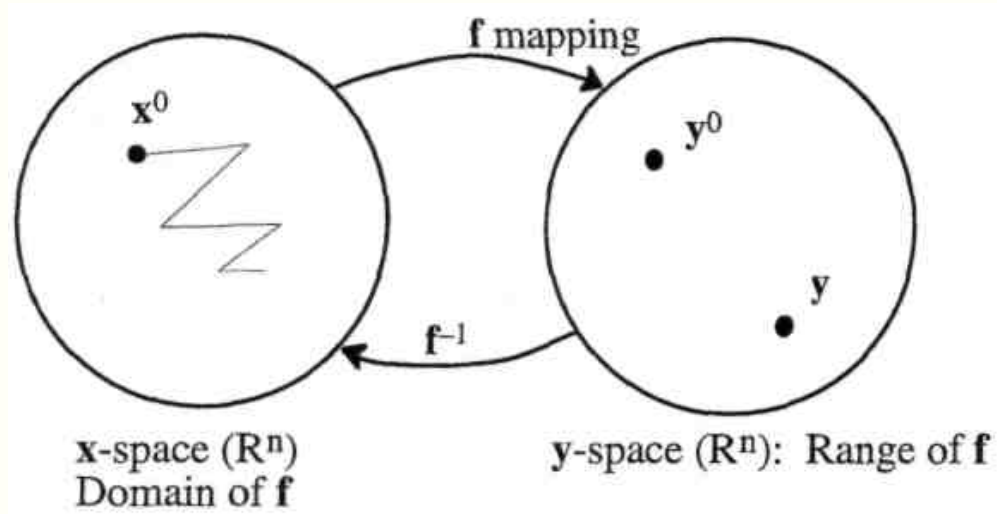
  or,

  $f_1(x_1,x_2, \ldots,x_n) = y_1$

  $f_2(x_1,x_2, \ldots,x_n) = y_2$

  :

  $f_n(x_1,x_2, \ldots,x_n) = y_n$



Problem

- Given $\mathbf{y} \in \mathbf{R}^n$, find $\mathbf{x}^* \in \mathbf{R}^n$, if it exists, such that $\mathbf{f}(\mathbf{x}^*) = \mathbf{y}$.

- **Some Theorems on the Existence and Uniqueness of Solutions of Nonlinear Resistive Networks.**

Definition

- Given a mapping f(.): $R^n \rightarrow R^n$.

- **f** ε C^1 means **f** is continuously differentiable; and

  f is C^1 diffeomorphism means that the inverse

  function $\mathbf{f}^{-1}$ exists, and is also of class C^1.

Palais's Theorem

- The *necessary* and *sufficient conditions* that the mapping **f(.)**: $R^n \rightarrow R^n$ to be a C^1 ***diffeomorphism*** of $\mathbf{R^n}$ onto itself are:

  (i) **f** is of class C^1

  (ii) $\det[\mathbf{J}] \triangleq \det\left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right] \neq 0$

  (iii) $\lim \|\mathbf{f(x)}\| \rightarrow \infty$ as $\|\mathbf{x}\| \rightarrow \infty$

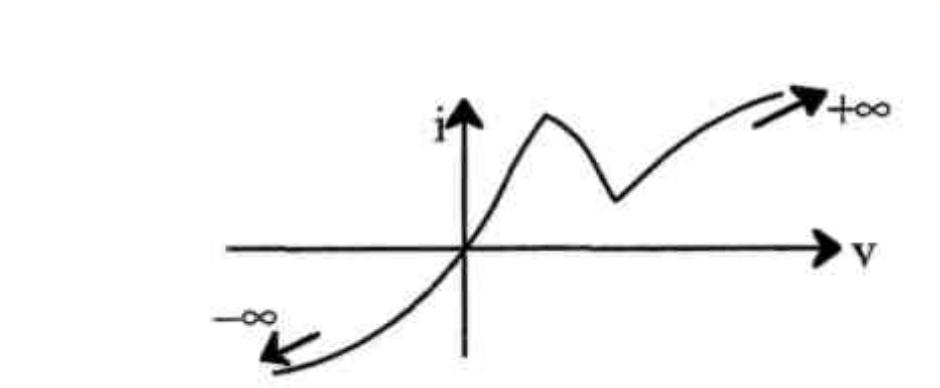
- For existence and uniqueness of solution, can allow det [**J**] = 0 at isolated points as long as it **does not change signs** and

  $\lim\|\mathbf{f(x)}\| \rightarrow \infty$ when $\|\mathbf{x}\| \rightarrow \infty$

Circuit-Theoretic Theorems

- **Theorem 2** (Duffin): In a network consisting of independent voltage and current sources, and voltage-controlled two-terminal resistors (**i = g(v)**), there exists *at least one solution* provided that each resistor's **v-i** characteristic function **g(v)** is continuous in **v** and satisfies:

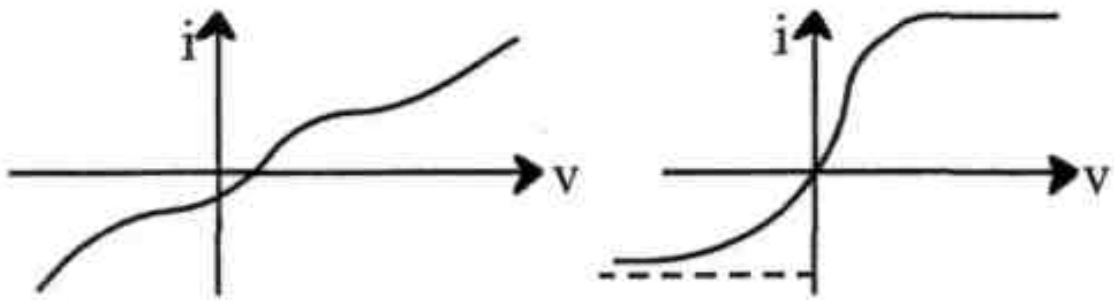
$$g(v) \rightarrow +\infty \text{ (or } -\infty)$$
$$\text{as } v \rightarrow +\infty \quad \text{(or } -\infty)$$



Circuit-Theoretic Theorems (cont.)

- **Theorem 3** (Duffin): In a network consisting of independent voltage and current sources and voltage-controlled resistors (**i = g(v)**), there exists *at most one solution* provided that each resistor's **v-i** characteristic function **g** is *strictly monotone increasing:*

$$g(v_1) < g(v_2) \quad \text{if} \quad v_1 < v_2$$



For existence and uniqueness, both Theorems 2 and 3 should be satisfied.

Circuit-Theoretic Theorems (cont.)

- **Theorem 3** (Desoer and Katzenelson): A **sufficient** condition for the existence of a unique solution for a network *consisting of time-varying voltage-controlled* and current-controlled resistors characterized by ***continuous*** (not necessarily strictly) monotone increasing functions, and *independent voltage* and *current sources,* is that the resistor network formed by *short-circuiting all voltage sources* and *open-circuiting all current sources* has a **tree** (or forest) such that all tree branches correspond to *current-controlled elements* and all links correspond to *voltage-controlled elements.*

Now back to the numerical solution of:

$$\mathbf{f(x) = y}$$

Given **y**, find **x** (assuming it exists)

Fixed-Point Iteration

$$\mathbf{x = g(x)}$$

•A given problem can be recast into fixed-point problem, where $\mathbf{x = g(x)}$ is a suitably chosen function whose solutions are the solution of $\mathbf{f(x) = y}$.

•For example, $\mathbf{f(x) = y}$ can be written as

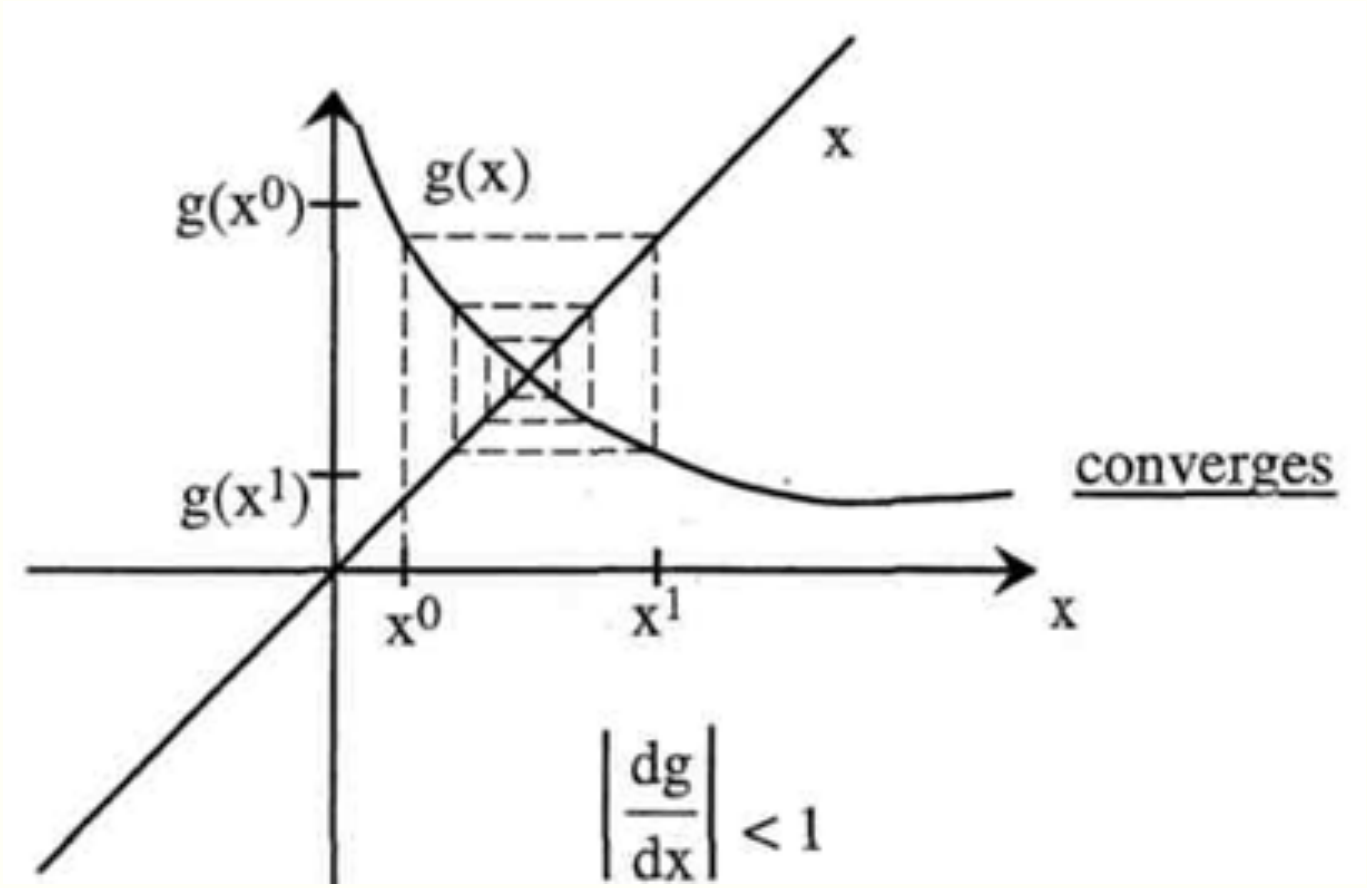
$$\mathbf{x = f(x) - y + x = g(x).}$$

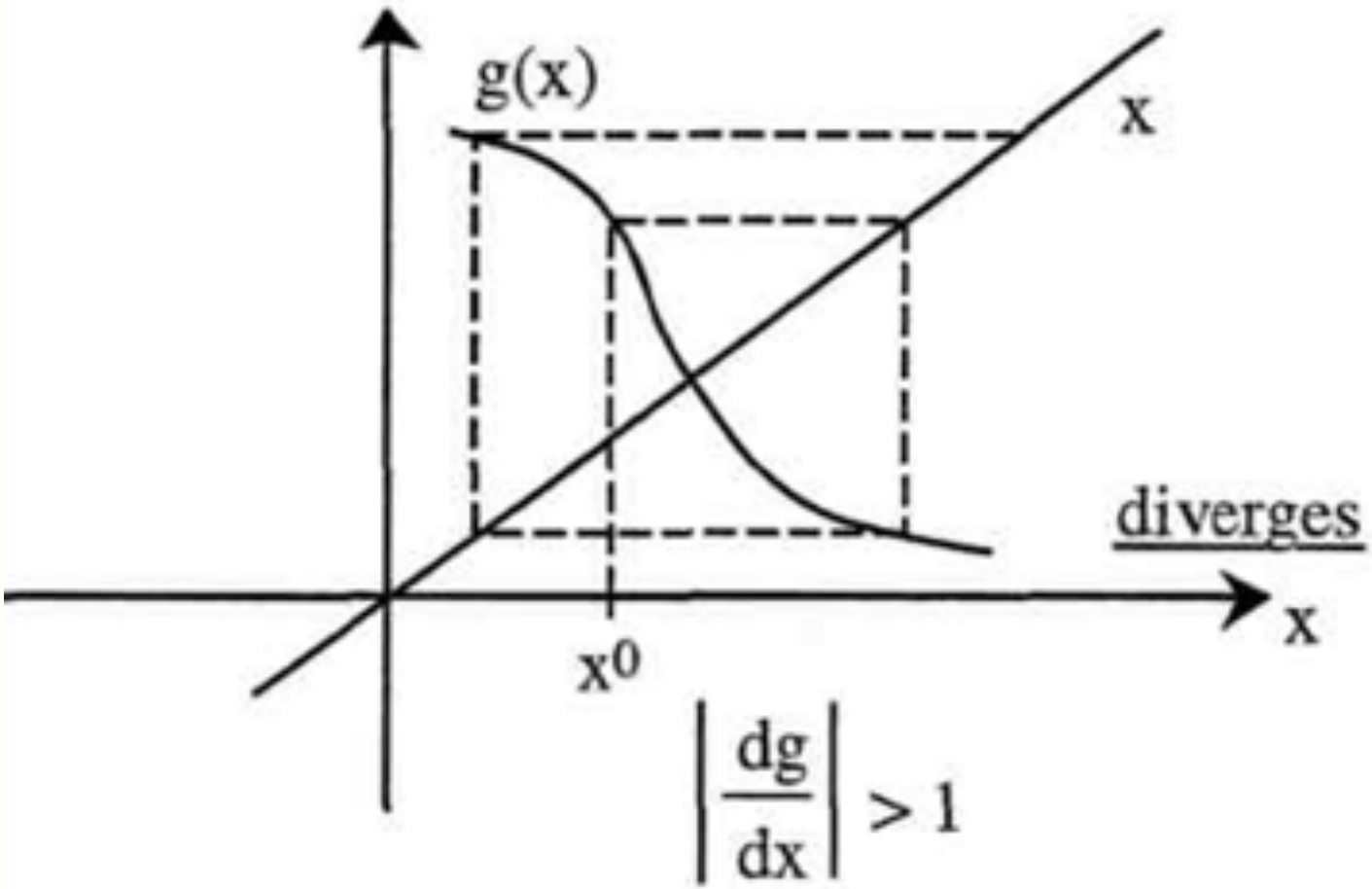
•Given $\mathbf{x = g(x)}$

•Fixed-Point Iteration: $\mathbf{x^{k+1} = g(x^k)}$

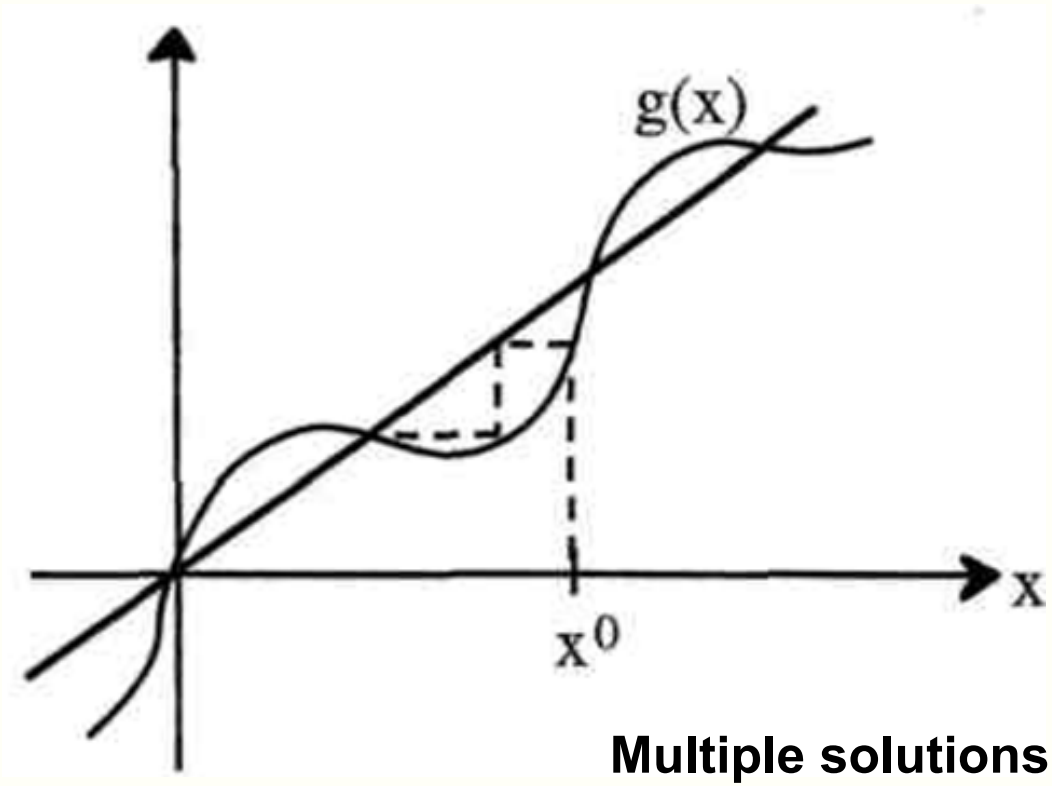
Repeat until $\mathbf{||x^{k+1} - x^k|| < \varepsilon}$

Examples




g(x)
x
diverges
x
x^0
$\left|\frac{dg}{dx}\right| > 1$

Examples (cont.)



Contraction mapping theorem

- Suppose **g:** $D \varepsilon R^n \rightarrow R^n$ maps a closed set $D_0 \varepsilon D$ into itself and $\|\mathbf{g(x)} - \mathbf{g(y)}\| \leq \alpha \|\mathbf{x-y}\|$, $\mathbf{x}, \mathbf{y} \varepsilon D_0$ for some $\alpha < 1$

  Then for any $\mathbf{x^0} \varepsilon D_0$, the *sequence* $\mathbf{x^{k+1} = g(x^k)}$, k = 0, 1,2, ..., converges to a *unique* $\mathbf{x^*}$ of **g** in $\mathbf{D_0}$.

- **Proof:**

  $$\|\mathbf{x^*-x^k}\| = \|\mathbf{g(x^*) - g(x^{k-1})}\|$$
  $$\leq \alpha \|\mathbf{x^*-x^{k-1}}\|$$
  $$\leq \alpha^k \|\mathbf{x^*-x^0}\|$$

  Since $\alpha < 1$, $\alpha^k \rightarrow 0$ and $\|\mathbf{x^* - x^k}\| \rightarrow 0$ or $\mathbf{x^k} \rightarrow \mathbf{x^*}$

Parallel Chord Method

A remains constant; it is usually chosen to be = $\left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{x = x^0}$ where

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \triangleq \begin{bmatrix} \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_2} & \cdots & \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}_n} \\ \frac{\partial \mathbf{f}_2}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{f}_2}{\partial \mathbf{x}_2} & \cdots & \frac{\partial \mathbf{f}_2}{\partial \mathbf{x}_n} \\ \vdots & & & \\ \frac{\partial \mathbf{f}_n}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{f}_n}{\partial \mathbf{x}_2} & \cdots & \frac{\partial \mathbf{f}_n}{\partial \mathbf{x}_n} \end{bmatrix}_{x=x^0}$$

"Jacobian matrix"

Instead of computing $\mathbf{A}^{-1}$, the following equation is solved:

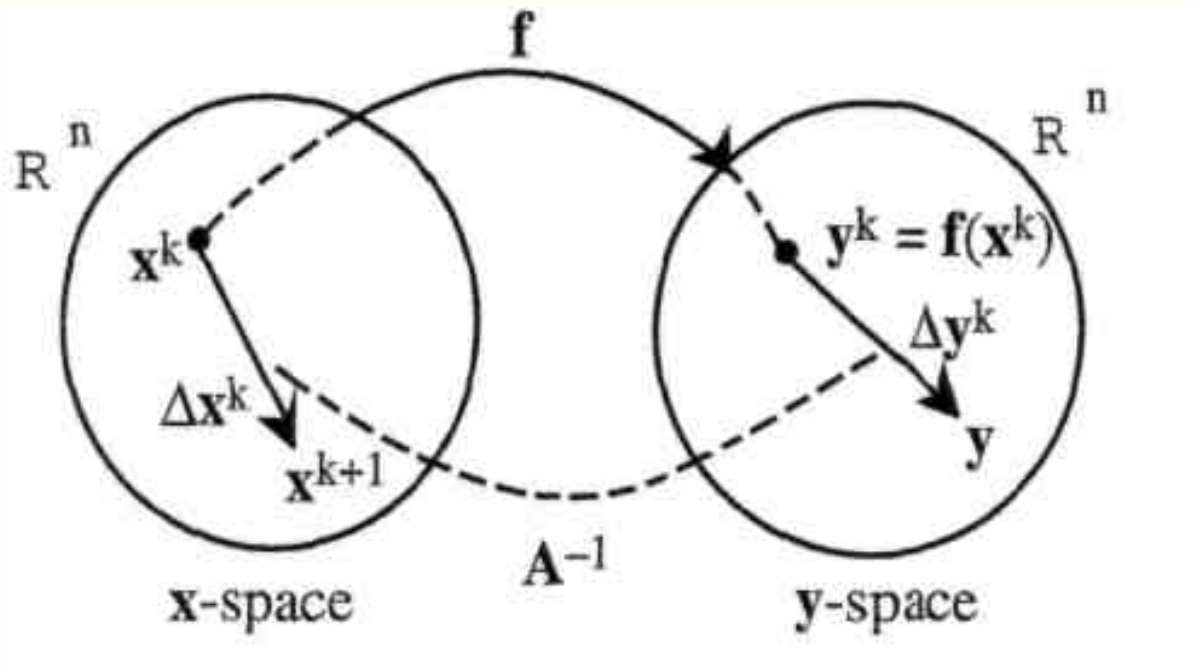
$$\mathbf{A}(\mathbf{x}^{k+1} - \mathbf{x}^k) = \mathbf{y} - \mathbf{f}(\mathbf{x}^k) \quad \text{or} \quad \mathbf{A}\ \Delta\mathbf{x}^k = \Delta\mathbf{y}^k$$

At every iteration, $\Delta\mathbf{y}^k$ changes, while **A** (and its **LU** factors) remain unchanged.

Parallel Chord Method

$$x^{k+1} = x^k + A^{-l}(y - f(x^k))$$

Or $$A(x^{k+1} - x^k) = (y - f(x^k))$$



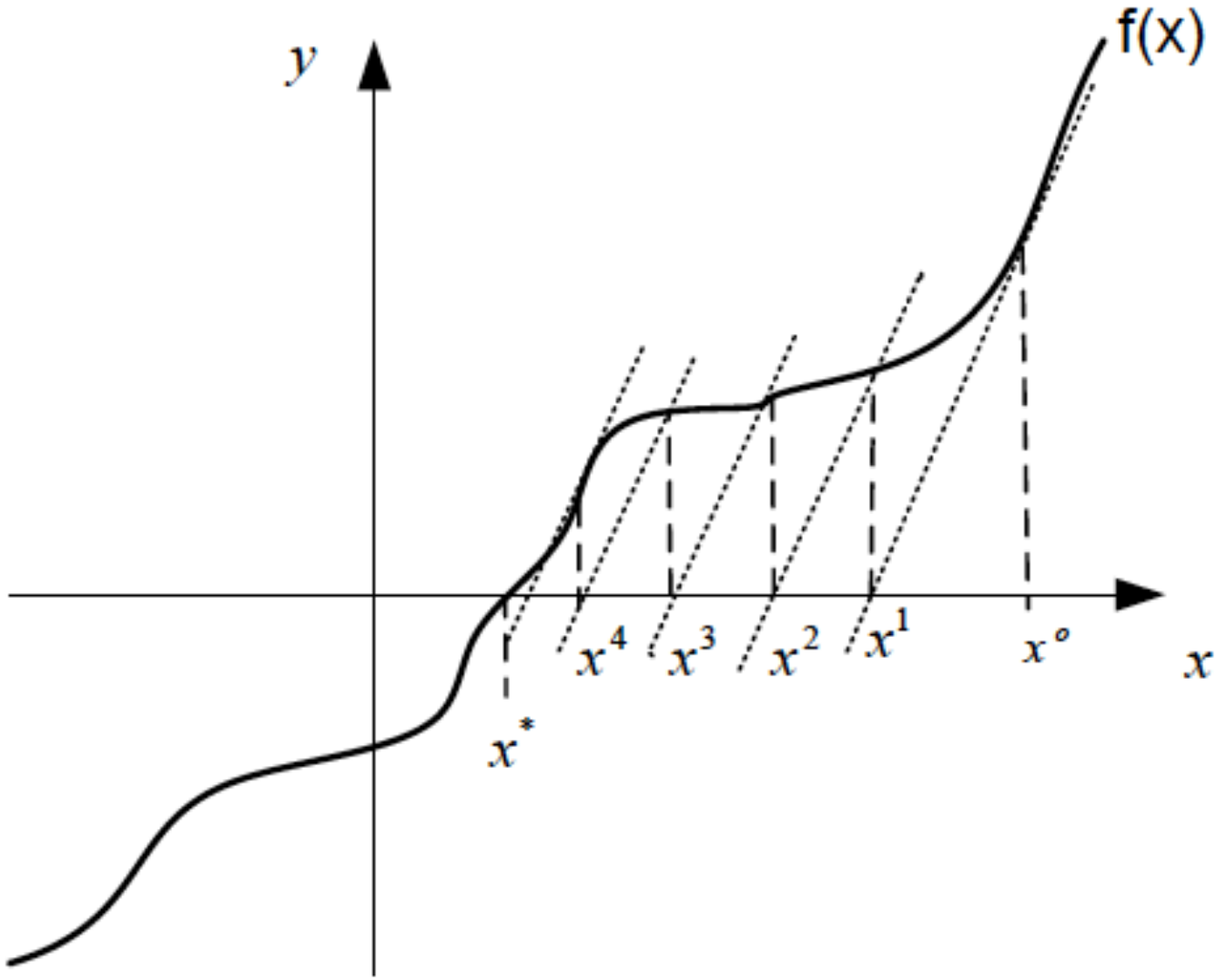
Parallel Chord Algorithm

Algorithm 6.2.2: Parallel Chord Iteration Algorithm

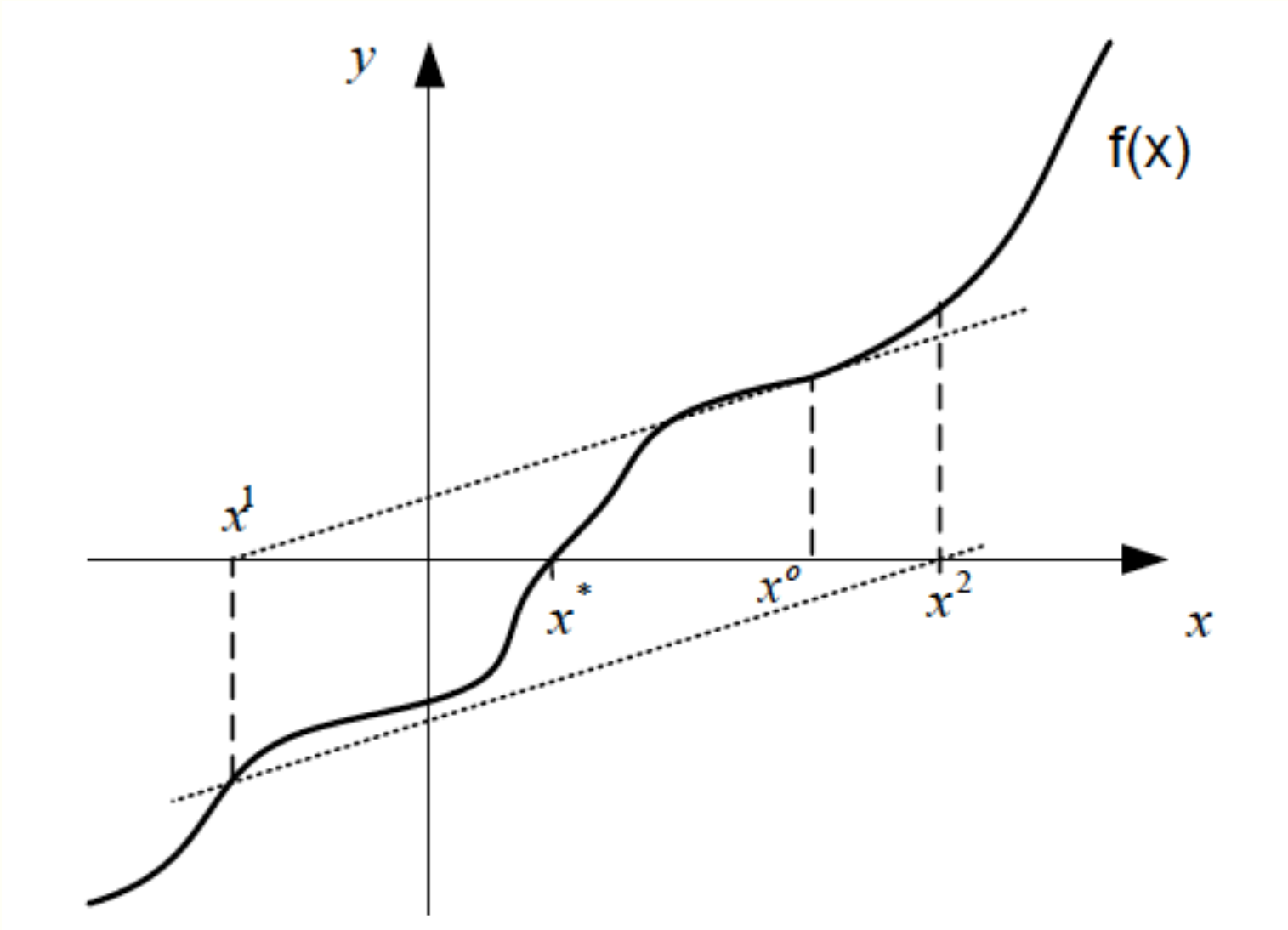
Given $\mathbf{f}(\mathbf{x}) = \mathbf{y}$ with a specified $\mathbf{y}$.

1. Start with an initial guess $\mathbf{x}^{(o)}$
2. Construct $\mathbf{A} = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]$ evaluated at $\mathbf{x} = \mathbf{x}^{(o)}$; or choose any other appropriate matrix $\mathbf{A}$
3. For $k = 0, 1, 2, \ldots\ldots$
4. Find $\triangle \mathbf{y}^{(k)} = \mathbf{y} - \mathbf{f}(\mathbf{x}^{(k)})$
5. Solve $\mathbf{A} \triangle \mathbf{x}^{(k)} = \triangle \mathbf{y}^{(k)}$
6. Put $\mathbf{x}^{(k)} = \mathbf{x}^{(k)} + \triangle \mathbf{x}^{(k)}$
7. Repeat until $\| \triangle \mathbf{x}^{(k)} \| < \varepsilon$ and $\| \triangle \mathbf{y}^{(k)} \| < \delta$, where ε and δ are specified small positive numbers.

Parallel Chord Method



Parallel Chord Example (nonconvergence)



Newton's (or Newton-Raphson) Method

Given: **f(x) = y**

$$f(x)=f(x^k) + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{x}^k}(x - x^k) = y$$

First two-terms of Taylor Series expansion

OR $[J^k](x-x^k) = (y-f(x^k))$

$[J^k]\,\Delta x^k = \Delta y^k$ (solve)

$x^{k+1} = x^k + \Delta x^k$

$$x^{k+1} = \underbrace{\mathbf{x}^k + \left[\mathbf{J}^k\right]^{-1}\left(\mathbf{y} - \mathbf{f}\left(\mathbf{x}^k\right)\right)}_{g\left(x^k\right)} = g(x^k)$$

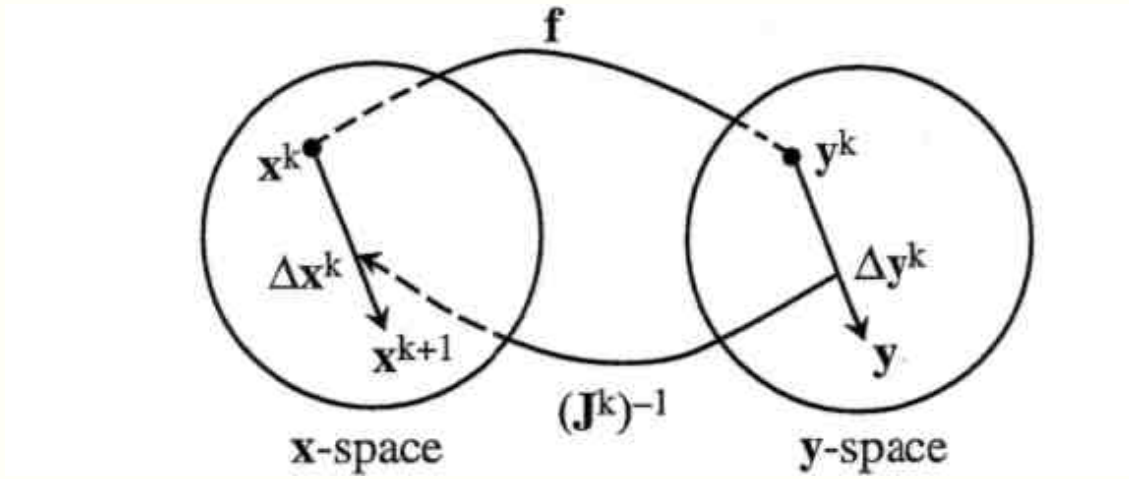
$$\text{where } J^k \equiv \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{x}^k}$$

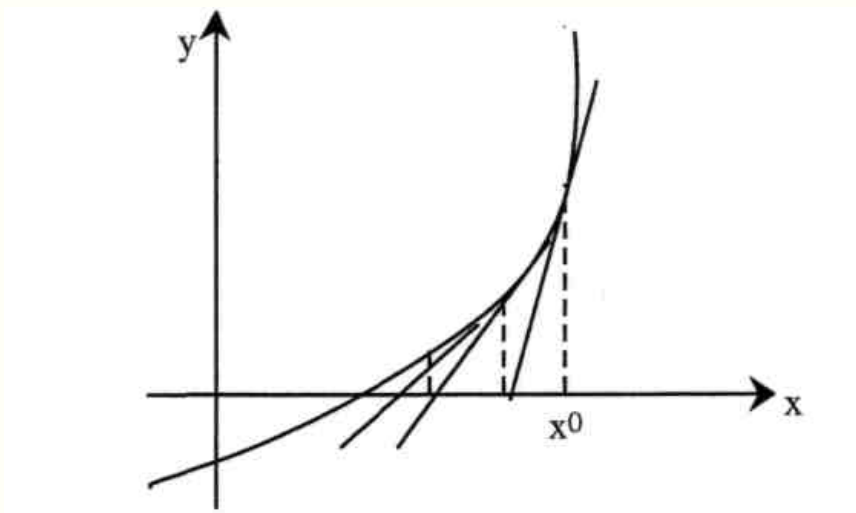
Newton or Newton-Raphson Algorithm

Given $\mathbf{f}(\mathbf{x}) = \mathbf{y}$ with a specified $\mathbf{y}$

1. Choose an initial guess $\mathbf{x}^{(\mathbf{o})}$
2. For $k = 0, 1, 2, \ldots\ldots$
3. Find $\triangle\mathbf{y}^{(k)} = \mathbf{y} - \mathbf{f}(\mathbf{x}^{(k)})$
4. Construct $\left[\mathbf{J}^{(k)}\right] = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]$ evaluated at $\mathbf{x} = \mathbf{x}^{(\mathbf{k})}$
5. Solve $\left[\mathbf{J}^{(k)}\right] \triangle\mathbf{x}^{(k)} = \triangle\mathbf{y}^{(k)}$
6. Put $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \triangle\mathbf{x}^{(k)}$
7. Repeat until $\| \triangle\mathbf{x}^{(k)} \| < \varepsilon$ and $\| \triangle\mathbf{y}^{(k)} \| < \delta$,

Newton's (or Newton-Raphson) Method





The slope $\frac{df}{dx}$ changes with every iteration.

Convergence Properties of Newton's Method

- Applying Taylor Series expansion at $\mathbf{x}^k$:

- $\mathbf{y} = \mathbf{f}(\mathbf{x}^*) = \mathbf{f}(\mathbf{x}^k) + J^k(\mathbf{x}^* - \mathbf{x}^k) + R(\mathbf{x}^* - \mathbf{x}^k)$

  where $\mathbf{x}^*$ is the solution

- If the derivative of $\mathbf{J}^k$ (i.e., second derivative of $\mathbf{f}$) is bounded, then:

$$\|R(\mathbf{x}^k - \mathbf{x}^*)\| \leq \alpha \, \|\mathbf{x}^k - \mathbf{x}^*\|^2$$

- Newton's Method:

$$\mathbf{x}^{k+1} = \mathbf{x}^k + [\mathbf{J}^k]^{-1} (\mathbf{y} - \mathbf{f}(\mathbf{x}^k))$$

  or, $[\mathbf{J}^k](\mathbf{x}^{k+1} - \mathbf{x}^k) = \mathbf{y} - \mathbf{f}(\mathbf{x}^k)$

n-dimensional case (cont.)

From Taylor Series:

- $\mathbf{y - f(x^k) = J^k(x^* - x^k) + R(x^* - x^k)}$
- $\mathbf{J^k(x^{k+l} - x^k) = J^k(x^* - x^k) + R(x^* - x^k)}$
- $\mathbf{x^{k+l} - x^k = x^* - x^k + [J^k]^{-1}R(x^* - x^k)}$
- $\mathbf{x^{k+l} - x^* = [J^k]^{-l}R(x^* - x^k)}$
- $\mathbf{||x^{k+1} - x^*|| \leq} c\ \mathbf{||x^* - x^k||^2}$

Provided $\mathbf{J(x^*)}$ is nonsingular

Rate of Convergence

- Define $\mathbf{e_k = x^* - x^k}$

- A method is said to converge with rate r if:

  $$\|\mathbf{e_{k+1}}\| = c\,\|\mathbf{e_k}\|^r$$

  for some nonzero constant c

- If r = 1, the convergence is **linear.**

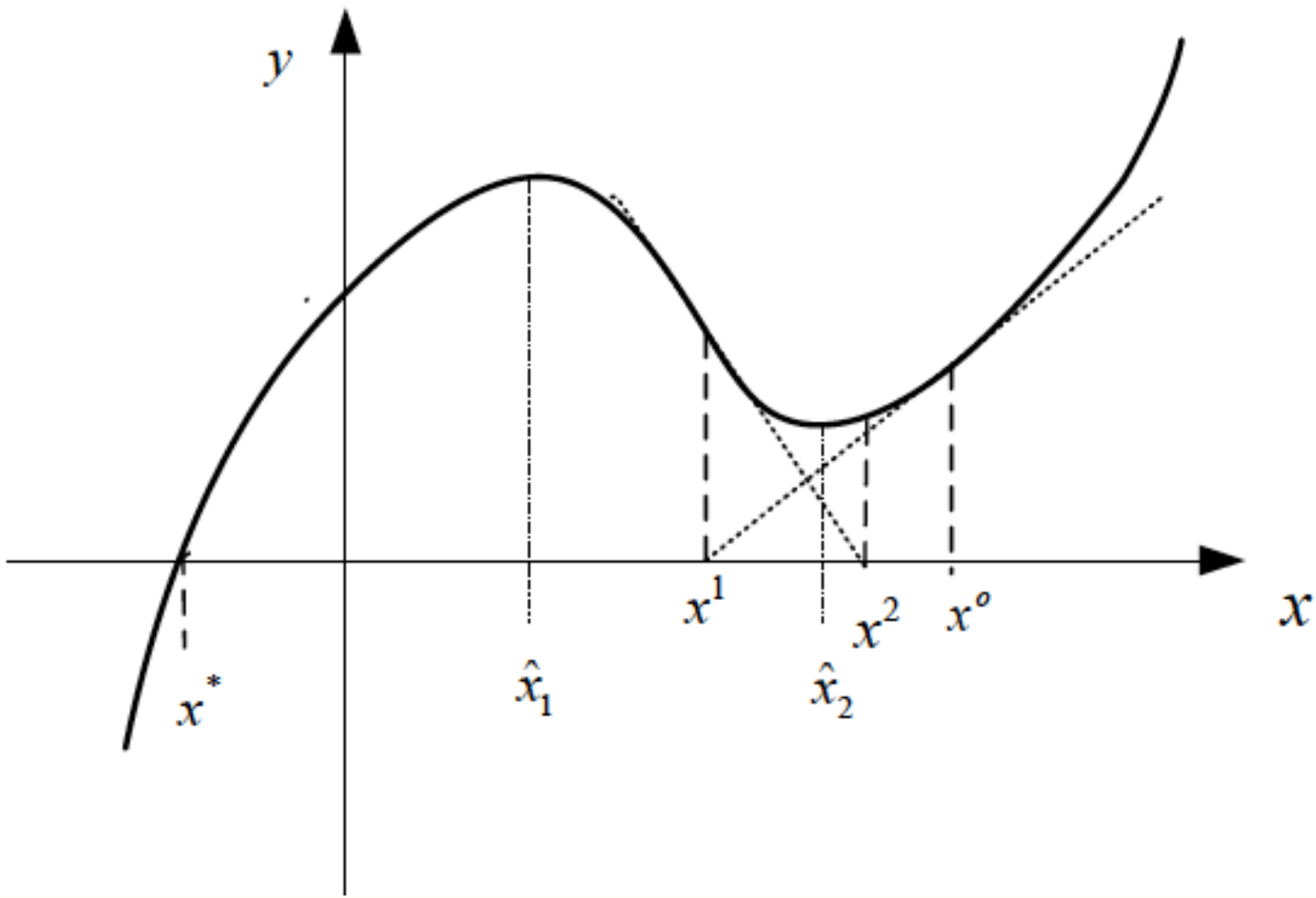
- If r > 1, the convergence rate is **superlinear.**

- If r = 2, the convergence rate is **quadratic.**

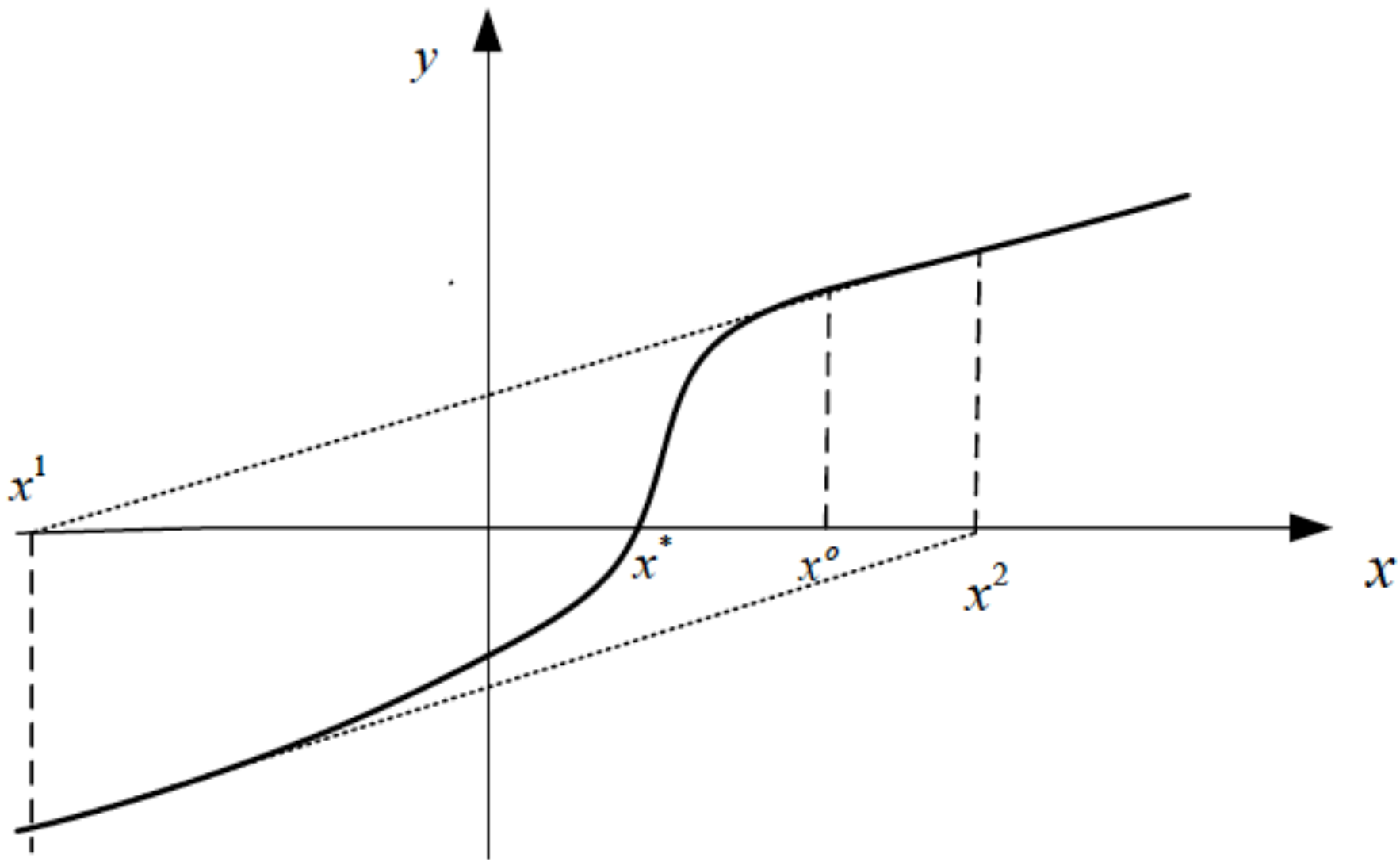
Newton's Method

Has *quadratic convergence* if $\mathbf{x}^k$ is "**close enough**" to the **solution** and $\mathbf{J(x^*)}$ is *nonsingular.*

Convergence Problems of Newton's Method



Convergence Problems of Newton's Method



Pseudo-Newton or Norm-Reducing Techniques

(1) In the Newton algorithm the Jacobian is constructed and factorized at every iteration, while in the parallel-chord the Jacobian is constructed and factorized once, and only the forward and backward substitutions are applied at every iteration. One strategy to reduce computation is to apply the parallel-chord algorithm as long as $\|\mathbf{y} - \mathbf{y}^{(k+1)}\| < \|\mathbf{y} - \mathbf{y}^{(k)}\|$ and switch to Newton method when (i) $\|\mathbf{y} - \mathbf{y}^{(k+1)}\| \geq \|\mathbf{y} - \mathbf{y}^{(k)}\|$; (ii) when the iterations approach the solution, that is, $\|\mathbf{y} - \mathbf{y}^{(k+1)}\| < \sigma$, for some $\sigma > \delta$; or (iii) when convergence slows down, since Newton algorithm has quadratic convergence in the vicinity of the solution, while the parallel-chord method has linear convergence.

(2) In Newton algorithm, after solving $\left[\mathbf{J}^{(k)}\right] \triangle\mathbf{x}^{(k)} = \triangle\mathbf{y}^{(k)}$, put

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \lambda\triangle\mathbf{x}^{(k)}$$

where λ is chosen such that $\|\mathbf{y} - \mathbf{y}^{(k+1)}\| < \|\mathbf{y} - \mathbf{y}^{(k)}\|$

This normally requires a line search

along the vector $\mathbf{x}^{(k)} + \lambda\triangle\mathbf{x}^{(k)}$

(3) After solving $\left[\mathbf{J}^{(k)}\right] \triangle \mathbf{x}^{(k)} = \triangle \mathbf{y}^{(k)}$, put

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix} \triangle \mathbf{x}^{(k)}$$

(4) When solving $\left[\mathbf{J}^{(k)}\right] \triangle\mathbf{x}^{(k)} = \triangle\mathbf{y}^{(k)}$, if $\mathbf{J}^{(k)}$ is singular or ill-conditioned, or if a zero or a small pivot is encountered and sparse matrix reordering is not desirable, one can solve a modified system

$$\left[\mathbf{J}^{(k)} + \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix}\right] \triangle\mathbf{x}^{(k)} = \triangle\mathbf{y}^{(k)}$$

Normally, very few of the λ_is are nonzero. To save on computation, it is possible to correct the solution of (6.2.17) to get the solution of the original equation $\left[\mathbf{J}^{(k)}\right] \triangle\mathbf{x}^{(k)} = \triangle\mathbf{y}^{(k)}$ by using the solution updating algorithm

(5) Source Stepping: If $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ is easy to solve, or if it is readily available, solve

$$\mathbf{f}(\mathbf{x}) = \lambda \mathbf{y}$$

for $\lambda = 0, \lambda_1, \lambda_2, \ldots, 1$, where $0 < \lambda_i < \lambda_{i+1} < 1$. The solution of the equations at λ_i is used as the initial guess for the solution at λ_{i+1}. When $\lambda = 1$, $\mathbf{f}(\mathbf{x}) = \mathbf{y}$. The philosophy behind this strategy is that the solution at λ_{i+1} is close to the solution at λ_i, if the increment in λ is small enough. In circuit application, source stepping is equivalent to putting initially all the independent sources to zero, which gives a zero solution, and then increasing the values of the independent sources in increments until the given values are reached.

(6) Homotopy: Solve

$$\lambda \mathbf{f}(\mathbf{x}) + (1-\lambda)(\mathbf{x} - \mathbf{x}^{(o)}) = \lambda \mathbf{y}$$

for $\lambda = 0, \lambda_1, \lambda_2, \ldots, 1$, where $0 < \lambda_i < \lambda_{i+1} < 1$. The solution of the equations at λ_i is used as the initial guess for the solution at λ_{i+1}. When $\lambda = 0$, $\mathbf{x} = \mathbf{x}^{(o)}$; when $\lambda = 1$, $\mathbf{f}(\mathbf{x}) = \mathbf{y}$. Again, the philosophy behind this strategy is that the solution at λ_{i+1} is close to the solution at λ_i, if the increment in λ is small enough.

(7) Approximate $\mathbf{f}(\mathbf{x})$ by another *close* function, $\hat{\mathbf{f}}$, that is easy to solve, such as a piecewise-linear function, then use the solution of $\hat{\mathbf{f}}(\mathbf{x}) = \mathbf{y}$ as the initial

solution to solve $\mathbf{f}(\mathbf{x}) = \mathbf{y}$

Piecewise-linear solution techniques are covered in Chapter 7

(8) Form a differential equation:

$$\dot{\mathbf{x}} + \mathbf{f}(\mathbf{x}) = \mathbf{y}u(t)$$

where $\dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dt}$, and $u(t)$ is the step function. Assume the initial condition at $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ can be easily found (when all inputs are put to zero, normally $\mathbf{x} = \mathbf{0}$). When $t \to \infty$, the steady-state solution of (6.2.20) approaches the solution of

$$\mathbf{f}(\mathbf{x}) = \mathbf{y}$$

The numerical solution of differential equations is covered in

Chapter 8

Application to Electronic Circuits:

- **Capacitors are open and inductors are short-circuited.** Why?

Tableau Equations:

$$\text{KCL}: \ \mathbf{A}\mathbf{i}_\mathbf{b} = \mathbf{0} \ (\text{or } \mathbf{Q}\mathbf{i}_\mathbf{b} = \mathbf{0})$$

$$\text{KVL}: \mathbf{v}_\mathbf{b} = \mathbf{A}^\mathbf{T}\mathbf{v}_\mathbf{n} \ (\text{or } \mathbf{B}\mathbf{v}_\mathbf{b} = \mathbf{0})$$

$$\mathbf{i}_1 = \mathbf{f}_1(\mathbf{v}_1, \mathbf{i}_2) + \mathbf{s}_1$$

$$\mathbf{v}_2 = \mathbf{f}_2(\mathbf{v}_1, \mathbf{i}_2) + \mathbf{s}_2$$

In general, the characteristics could be of the form:

$$\mathbf{f}(\mathbf{v}_\mathbf{b}, \mathbf{i}_\mathbf{b}, \mathbf{z}) = \mathbf{0}$$

there is no need to formulate the nonlinear MNA equations since the solution is the same regardless of the formulation method used

The nonlinear elements are linearized at the tableau equation level

that is, at the element characteristics level

The linearized circuit equations are then formulated and solved using the MNA formulation

or any other formulation method

DC Solution Algorithm

1. Read the circuit topology specifications and the element characteristics.

2. Specify an initial solution guess, $\mathbf{x}^{(k)}$, $k = 0$. If the modified nodal analysis (MNA) formulation is used, then $\mathbf{x} = \begin{bmatrix} \mathbf{v_n} & \mathbf{i_2} \end{bmatrix}^{\mathrm{T}}$; if the extended nodal analysis (ENA) formulation is used, then $\mathbf{x} = \begin{bmatrix} \mathbf{v_n} & \mathbf{i_2} & \mathbf{z} \end{bmatrix}^{\mathrm{T}}$.

3. Linearize the nonlinear element characteristics at $\mathbf{x} = \mathbf{x}^{(k)}$.

4. Formulate the linearized circuit equations using any general formulation method, such as the MNA stamp formulation method, to get the linearized circuit equation either in the form

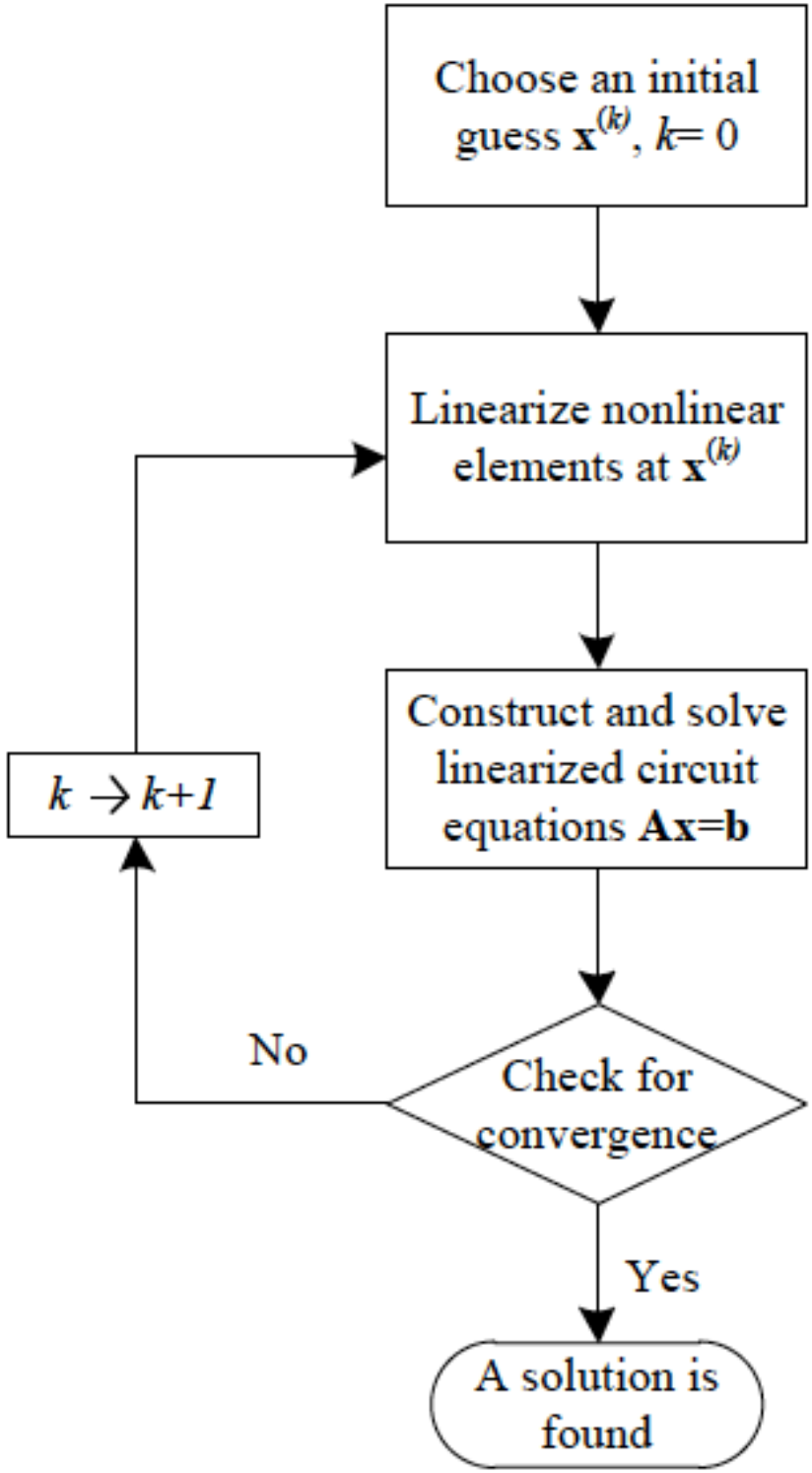
$$\mathbf{J}^{(k)}\mathbf{x} = \mathbf{b}^{(k)}, \text{ where } \mathbf{b}^{(k)} = \mathbf{y} - \mathbf{f}(\mathbf{x}^{(k)}) + \left[\mathbf{J}^{(k)}\right]\mathbf{x}^{(k)}$$

or as

$$\mathbf{J}^{(k)}\triangle\mathbf{x}^{(k)} = \triangle\mathbf{y}^{(k)}, \text{ where } \triangle\mathbf{y}^{(k)} = \mathbf{y} - \mathbf{f}(\mathbf{x}^{(k)})$$

5. Solve the linearized circuit equations, $\mathbf{J}^{(k)}\mathbf{x} = \mathbf{b}^{(k)}$ or $\mathbf{J}^{(k)}\triangle\mathbf{x}^{(k)} = \triangle\mathbf{y}^{(k)}$, normally by using **LU** factorization and sparse matrix techniques, or by an iterative method, to obtain $\mathbf{x}^{(k+1)}$ or $\triangle\mathbf{x}^{(k)}$. The structure of $\mathbf{J}^{(k)}$ does not normally change from one iteration to the next, but the values of its entries may change.

6. *Check for convergence*: Check if $\|\triangle\mathbf{x}^{(k)}\| = \|\mathbf{x}^{(k)} - \mathbf{x}^{(k+1)}\| < \epsilon$ (in circuit simulation normally the $\mathcal{L}_\infty$-norm is used). If not, go to step 3 and repeat. If $\|\triangle\mathbf{x}^{(k)}\| < \epsilon$, check if $\|\triangle\mathbf{y}^{(k+1)}\| = \| \mathbf{y} - \mathbf{y}^{(k+1)} \| < \delta$. If it is, a solution is found. If not, a *false* solution is reached, and a new initial point should be chosen, or a modified Newton method applied (see Figure 6.2.9). In some applications, $\triangle\mathbf{y}^{(k+1)}$ is not computed to save on computation, and a solution is declared when $\|\triangle\mathbf{x}^{(k)}\| < \epsilon$, even though a solution is not guaranteed to have been reached.


Choose an initial guess $\mathbf{x}^{(k)}$, k= 0
Linearize nonlinear elements at $\mathbf{x}^{(k)}$
Construct and solve linearized circuit equations **Ax=b**
Check for convergence
No
$k \rightarrow k+1$
Yes
A solution is found

- In step (5) above, a relaxation method may be applied to solve $\mathbf{Ax} = \mathbf{b}$ rather than $\mathbf{LU}$ factorization.

- If the parallel-chord method is applied, only the right-hand side vector $\mathbf{b}$ is constructed at the next iteration and the $\mathbf{LU}$ factorization is not performed in step 5; only the forward and backward substitution steps are repeated.

- In step 3, only the nonlinear resistors in which the independent variables did not converge are re-linearized.

- Modified Newton methods described in the previous section can be made part of the above algorithm without much difficulty.

- It is possible to mix the parallel-chord method with the Newton method. For example, one could use the parallel-chord method as long as the iterates are converging in order to save on formulating the linearized equations and performing LU factorization at every iteration, then switch to Newton method when the iterates approach the solution, that is, $\|\triangle \mathbf{x}^{(k)}\|$ is less than a certain threshold, or when the iterates diverge.

Stamps: Nonlinear Elements

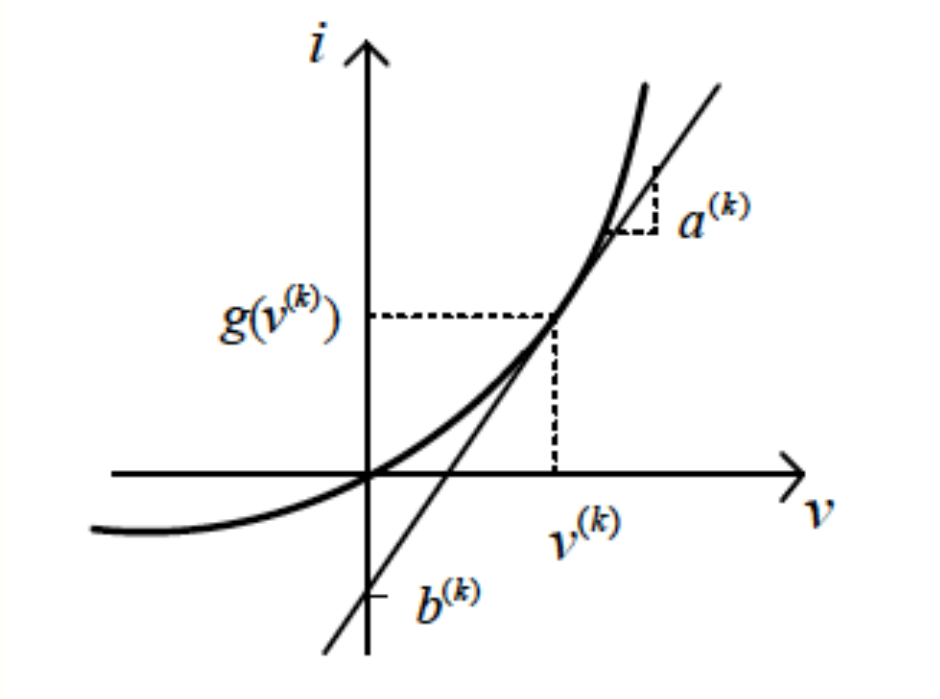
Two-terminal voltage-controlled resistor:

$$i = g(v)$$

Taylor series expansion at an iteration point $v^{(k)}$

$$i = g(v^{(k)}) + \left[\frac{dg}{dv}\right]_{v=v^{(k)}} (v - v^{(k)}) = a^{(k)}v + g(v^{(k)}) - a^{(k)}v^{(k)}$$

$$i = a^{(k)}v + b^{(k)}$$



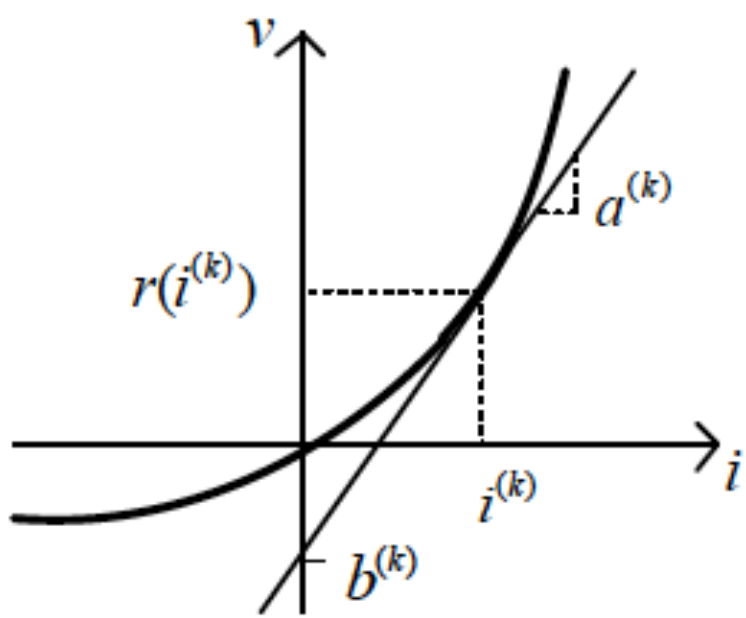
$$
\begin{array}{c}
 \\ n_1 \\ \\ n_2 \\ \\
\end{array}
\begin{array}{c}
\begin{array}{ccccc} & n_1 & & n_2 & \end{array} \\
\begin{bmatrix}
 & \vdots & & \vdots & \\
\cdots & +a^{(k)} & \cdots & -a^{(k)} & \cdots \\
 & \vdots & & \vdots & \\
\cdots & -a^{(k)} & \cdots & +a^{(k)} & \cdots \\
 & \vdots & & \vdots &
\end{bmatrix}
\end{array}
\begin{bmatrix} \\ v_{n1} \\ \\ v_{n2} \\ \\ \end{bmatrix},
\begin{bmatrix} rhs \\ \\ -b^{(k)} \\ \\ +b^{(k)} \\ \\ \end{bmatrix}
$$

Two-terminal current-controlled resistor:

$$v = r(i)$$

$$v = r(i^{(k)}) + \left[\frac{dr}{di}\right]_{i=i^{(k)}} (i - i^{(k)}) = a^{(k)} i + r(i^{(k)}) - a^{(k)} i^{(k)}$$

$$v = a^{(k)} i + b^{(k)}$$

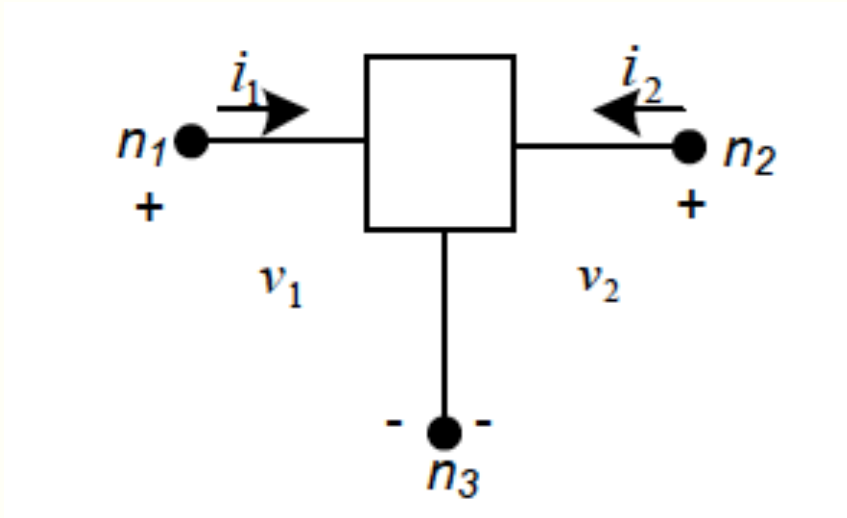


Two-terminal current-controlled resistor:

$$
\begin{array}{c}
 \\ n_1 \\ \\ n_2 \\ \\ \\
\end{array}
\begin{array}{c}
\begin{array}{cccccc} & n_1 & & n_2 & & \end{array} \\
\begin{bmatrix}
\cdots & & \cdots & & \cdots & +1 \\
 & \vdots & & \vdots & & \\
\cdots & & \cdots & & \cdots & -1 \\
 & \vdots & & \vdots & & \\
 & +1 & & -1 & & -a^{(k)}
\end{bmatrix}
\end{array}
\begin{bmatrix} \\ v_{n1} \\ \\ v_{n2} \\ \\ i \\ \end{bmatrix},
\begin{bmatrix} rhs \\ \\ \\ \\ +b^{(k)} \end{bmatrix}
$$

Stamp

Three-terminal voltage-controlled resistor:



$$i_1 = f_1(v_1, v_2)$$

$$i_2 = f_2(v_1, v_2)$$

Taylor series expansion

$$i_1 = a_{11}^{(k)} v_1 + a_{12}^{(k)} v_2 + b_1^{(k)}$$

$$i_2 = a_{21}^{(k)} v_1 + a_{22}^{(k)} v_2 + b_2^{(k)}$$

$$a_{11}^{(k)} = \frac{\partial f_1}{\partial v_1}, \quad a_{12}^{(k)} = \frac{\partial f_1}{\partial v_2}, \quad a_{21}^{(k)} = \frac{\partial f_2}{\partial v_1}, \quad a_{22}^{(k)} = \frac{\partial f_2}{\partial v_2}$$

evaluated at $v_1 = v_1^{(k)}, \; v_2 = v_2^{(k)}$

$$b_1^{(k)} = f_1(v_1^{(k)}, v_2^{(k)}) - (a_{11}^{(k)} v_1^{(k)} + a_{12}^{(k)} v_2^{(k)}),$$

$$b_2^{(k)} = f_2(v_1^{(k)}, v_2^{(k)}) - (a_{21}^{(k)} v_1^{(k)} + a_{22}^{(k)} v_2^{(k)})$$

Three-terminal voltage-controlled resistor:

$$\begin{bmatrix} n_1 & n_2 & n_3 \\ +a_{11}^{(k)} & +a_{12}^{(k)} & -(a_{11}^{(k)} + a_{12}^{(k)}) \\ +a_{21}^{(k)} & +a_{22}^{(k)} & -(a_{21}^{(k)} + a_{22}^{(k)}) \\ -(a_{11}^{(k)} + a_{21}^{(k)}) & -(a_{12}^{(k)} + a_{22}^{(k)}) & a_{33}^{(k)} \end{bmatrix} \begin{bmatrix} \\ v_{n1} \\ v_{n2} \\ v_{n3} \end{bmatrix}, \begin{bmatrix} rhs \\ -b_1^{(k)} \\ -b_2^{(k)} \\ +b_1^{(k)} + b_2^{(k)} \end{bmatrix}$$

where $a_{33}^{(k)} = (a_{11}^{(k)} + a_{21}^{(k)} + a_{12}^{(k)} + a_{22}^{(k)})$

Three-terminal resistor with hybrid characteristics:

$$i_1 = f_1(v_1, i_2)$$

$$v_2 = f_2(v_1, i_2)$$

Applying Taylor series expansion

$$i_1 = a_{11}^{(k)} v_1 + a_{12}^{(k)} i_2 + b_1^{(k)}$$

$$v_2 = a_{21}^{(k)} v_1 + a_{22}^{(k)} i_2 + b_2^{(k)}$$

Three-terminal resistor with hybrid characteristics:

$$\begin{bmatrix} n_1 & n_2 & n_3 & \vdots & \\ +a_{11}^{(k)} & & -a_{11}^{(k)} & \vdots & +a_{12}^{(k)} \\ & & & \vdots & +1 \\ -a_{11}^{(k)} & & +a_{11}^{(k)} & \vdots & (-1-a_{12}^{(k)}) \\ \dots & \dots & \dots & \vdots & \dots \\ -a_{21}^{(k)} & +1 & (-1+a_{21}^{(k)}) & \vdots & -a_{22}^{(k)} \end{bmatrix} \begin{bmatrix} v_{n1} \\ v_{n2} \\ v_{n3} \\ \dots \\ i_2 \end{bmatrix}, \begin{bmatrix} rhs \\ -b_1^{(k)} \\ \\ +b_1^{(k)} \\ \dots \\ +b_2^{(k)} \end{bmatrix}$$

Other methods of linearization

Approximate differentiation:

$$\frac{df}{dx} \approx \frac{f(x^{(k)} + \triangle x^{(k)}) - f(x^{(k)})}{\triangle x^{(k)}} = a^{(k)}$$

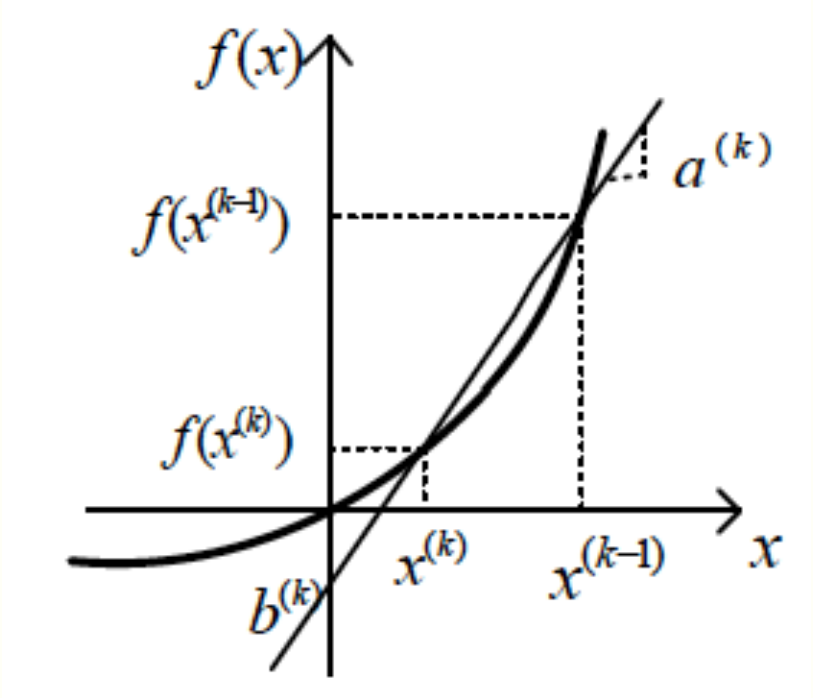
$$y = a^{(k)}x + b^{(k)} \qquad b^{(k)} = f(x^{(k)}) - a^{(k)}x^{(k)}$$

$$\mathbf{y} = \mathbf{f}(\mathbf{x})$$

$$\frac{\partial f_i}{\partial x_j} \approx \frac{f_i(x_1^{(k)}, \cdots, x_j^{(k)} + \triangle x_j^{(k)}, \cdots, x_n^{(k)}) - f_i(\mathbf{x}^{(k)})}{\triangle x_j^{(k)}}$$

$$\mathbf{y} = \mathbf{J}^{(k)}\mathbf{x} + \mathbf{b}^{(k)}, \ \mathbf{b}^{(k)} = (\mathbf{f}(\mathbf{x}^{(k)}) - \mathbf{J}^{(k)}\mathbf{x}^{(k)})$$

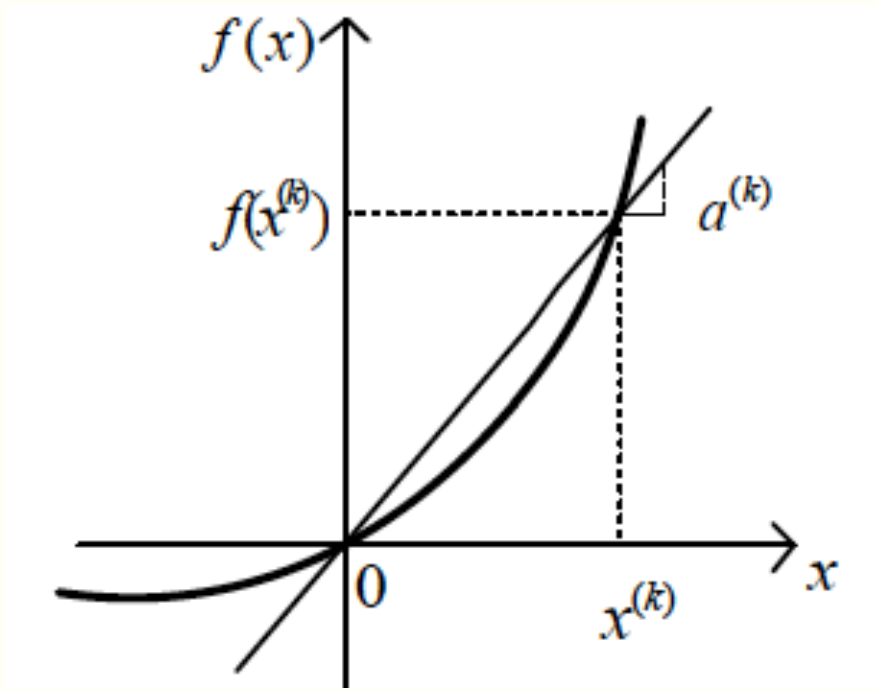
(2) Secant method:



$$\frac{df}{dx} \approx \frac{f(x^{(k)}) - f(x^{(k-1)})}{x^k - x^{k-1}} = a^{(k)}$$

$$\frac{\partial f_i}{\partial x_j} \approx \frac{f_i(\mathbf{x}^{(k)}) - f_i(x_1^{(k)}, \cdots, x_j^{(k-1)}, \cdots, x_n^{(k)})}{x_j^{(k)} - x_j^{(k-1)}}$$

(3) Line-through-the-origin:



$a^{(k)} = \frac{f(x^{(k)})}{x^{(k)}}$ and $b^{(k)} = 0$,

n-dimensional case $\frac{\partial f_i}{\partial x_j} \approx f_i(\mathbf{x}^{(k)})/x_j$, and $\mathbf{b}^{(k)} = \mathbf{0}$

Diode:

The characteristic equation of a diode, not including breakdown

$$i_d = g(v_d) = I_s(e^{v_d/\eta V_T} - 1)$$

typical value for I_s is 10^{-14}A

$V_T \approx 26$ mV at 300°K

Taylor series expansion an iteration point v_d^k

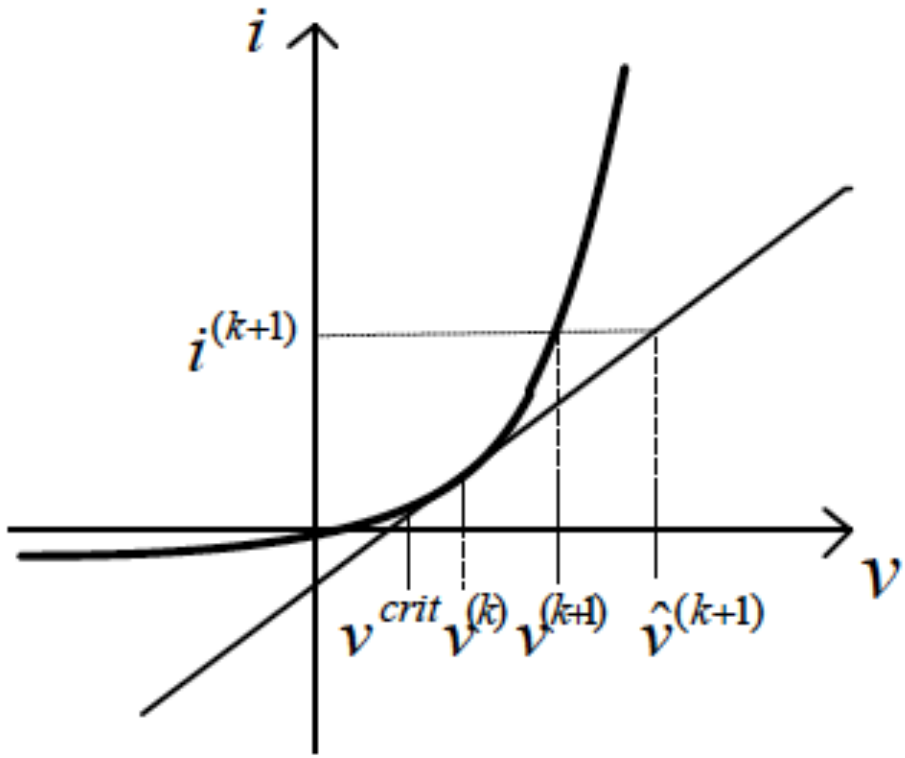
$$i_d = a^{(k)} v_d + b^{(k)}$$

$$a^{(k)} = \frac{I_s}{\eta V_T} e^{v_d^{(k)}/\eta V_T} \text{ and } b^{(k)} = g(v_d^{(k)}) - a^{(k)} v_d^{(k)}$$

$$
\begin{array}{c}
\\
n_1 \\
n_2 \\
\\
\end{array}
\begin{array}{c}
\begin{array}{ccccc} & n_1 & & n_2 & \end{array} \\
\begin{bmatrix}
 & \vdots & & \vdots & \\
\cdots & +a^{(k)} & \cdots & -a^{(k)} & \cdots \\
 & \vdots & & \vdots & \\
\cdots & -a^{(k)} & \cdots & +a^{(k)} & \cdots \\
 & \vdots & & \vdots &
\end{bmatrix}
\end{array}
\begin{bmatrix} \\ v_{n1} \\ v_{n2} \\ \\ \end{bmatrix},
\begin{bmatrix} rhs \\ -b^{(k)} \\ +b^{(k)} \\ \\ \end{bmatrix}
$$

if $v_d^{(k)} = 5\text{V}$ $\quad g(v_d^{(k)}) \approx 3.272\times10^{69}$ and $a^{(k)} \approx 1.2585\times10^{72}$

Voltage-Current Iteration Strategy



Newton iteration produced a new iteration point

$$\hat{v}_d^{(k+1)} = v_d^{(k)} + \triangle v_d^{(k)}$$

(a) If $\hat{v}_d^{(k+1)} \leq v_d^{crit}$ $\qquad v_d^{(k+1)} = \hat{v}_d^{(k+1)}$

(b) If $\hat{v}_d^{(k+1)} > v_d^{crit}$

$$\hat{i}_d^{(k+1)} = a^{(k)} \hat{v}_d^{(k+1)} + b^{(k)}$$

$$\hat{i}_d^{(k+1)} = I_s(e^{v_d^{(k+1)}/\eta V_T} - 1), \text{ or}$$

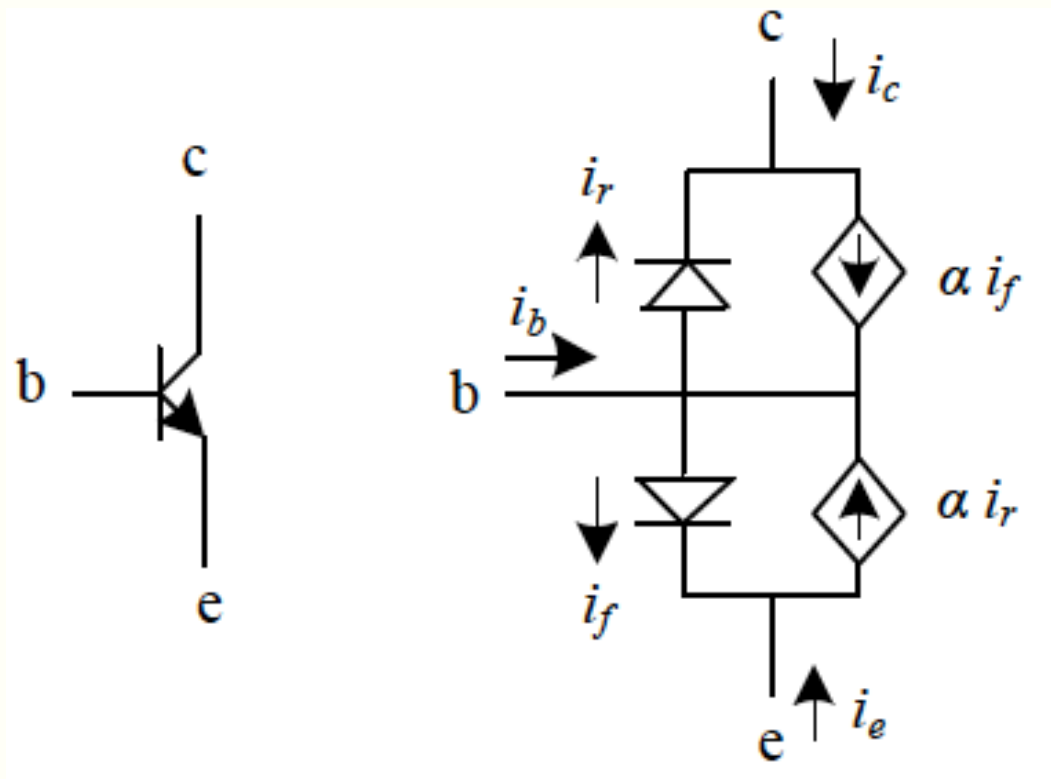
$$v_d^{(k+1)} = \eta V_T ln(1 + \frac{\hat{i}_d^{(k+1)}}{I_s})$$

$$v_d^{crit} = \eta V_T ln(\frac{\eta V_T}{\sqrt{2} I_s})$$

v_d^{crit} is the value of v_d where the *radius of curvature* of the diode characteristic equation is minimum

If $\eta V_T = 26$ mV and $I_s = 10^{-14}$, then $v_d^{crit} \approx 0.734$V

NPN Bipolar Junction Transistor: Ebers-Moll Model



$$i_c = -i_r + \alpha_f i_f = -I_{cs}(e^{v_{bc}/V_{Tc}} - 1) + \alpha_f I_{es}(e^{v_{be}/V_{Te}} - 1)$$

$$i_e = -i_f + \alpha_r i_r = \alpha_r I_{cs}(e^{v_{bc}/V_{Tc}} - 1) - I_{es}(e^{v_{be}/V_{Te}} - 1)$$

Taylor series expansion at iteration point $v_{bc}^{(k)}$ and $v_{be}^{(k)}$

$$i_c = a_{11}^{(k)} v_{bc} + a_{12}^{(k)} v_{be} + b_1^{(k)}$$

$$i_e = a_{21}^{(k)} v_{bc} + a_{22}^{(k)} v_{be} + b_2^{(k)}$$

$$a_{11}^{(k)} = -\frac{I_{cs}}{V_{Tc}}(e^{v_{bc}^{(k)}/V_{Tc}});\ a_{12}^{(k)} = \frac{\alpha_f I_{es}}{V_{Te}}(e^{v_{be}^{(k)}/V_{Te}})$$

$$b_1^{(k)} = -I_{cs}(e^{v_{bc}^{(k)}/V_{Tc}} - 1) + \alpha_f I_{es}(e^{v_{be}^{(k)}/V_{Te}} - 1) - a_{11}^{(k)} v_{bc}^{(k)} - a_{12}^{(k)} v_{be}^{(k)};$$

$$a_{21}^{(k)} = \frac{\alpha_r I_{cs}}{V_{Tc}}(e^{v_{bc}^{(k)}/V_{Tc}});\ a_{22}^{(k)} = -\frac{I_{es}}{V_{Te}}(e^{v_{be}^{(k)}/V_{Te}});\ \text{and}$$

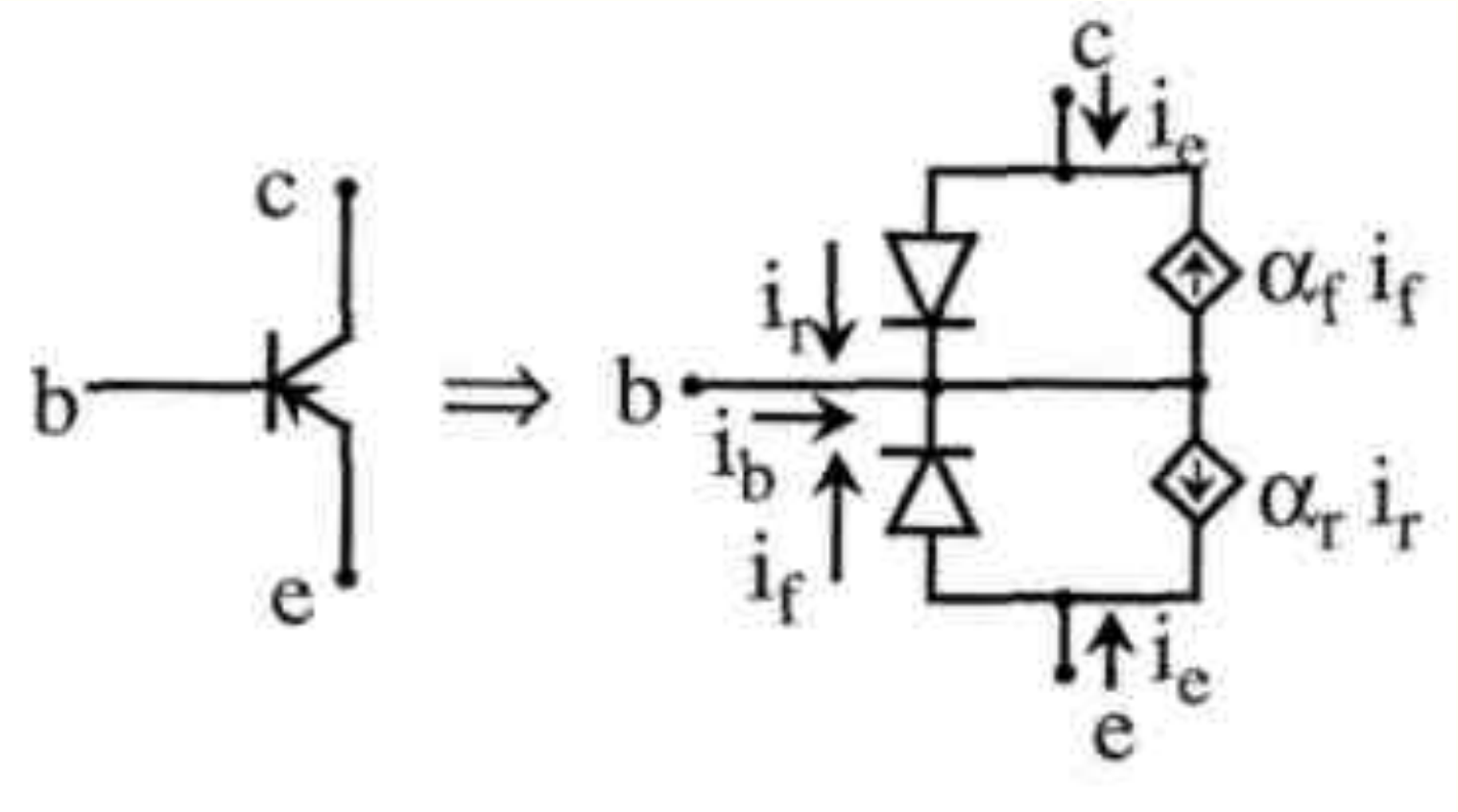
$$b_2^{(k)} = \alpha_r I_{cs}(e^{v_{bc}^{(k)}/V_{Tc}} - 1) - I_{es}(e^{v_{be}^{(k)}/V_{Te}} - 1) - a_{21}^{(k)} v_{bc}^{(k)} - a_{22}^{(k)} v_{be}^{(k)}.$$

Stamp

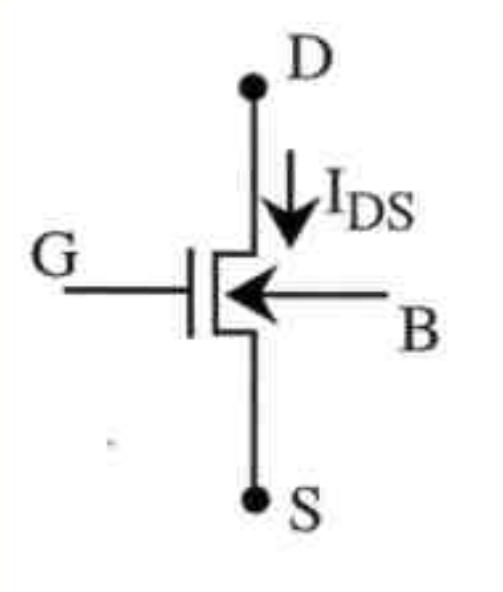
$$\begin{bmatrix} -a_{11}^{(k)} & -a_{12}^{(k)} & (a_{11}^{(k)} + a_{12}^{(k)}) \\ -a_{21}^{(k)} & -a_{22}^{(k)} & (a_{21}^{(k)} + a_{22}^{(k)}) \\ (a_{11}^{(k)} + a_{21}^{(k)}) & (a_{12}^{(k)} + a_{22}^{(k)}) & -a_{33}^{(k)} \end{bmatrix} \begin{bmatrix} v_c \\ v_e \\ v_b \end{bmatrix}, \begin{bmatrix} -b_1^{(k)} \\ -b_2^{(k)} \\ +b_1^{(k)} + b_2^{(k)} \end{bmatrix}$$

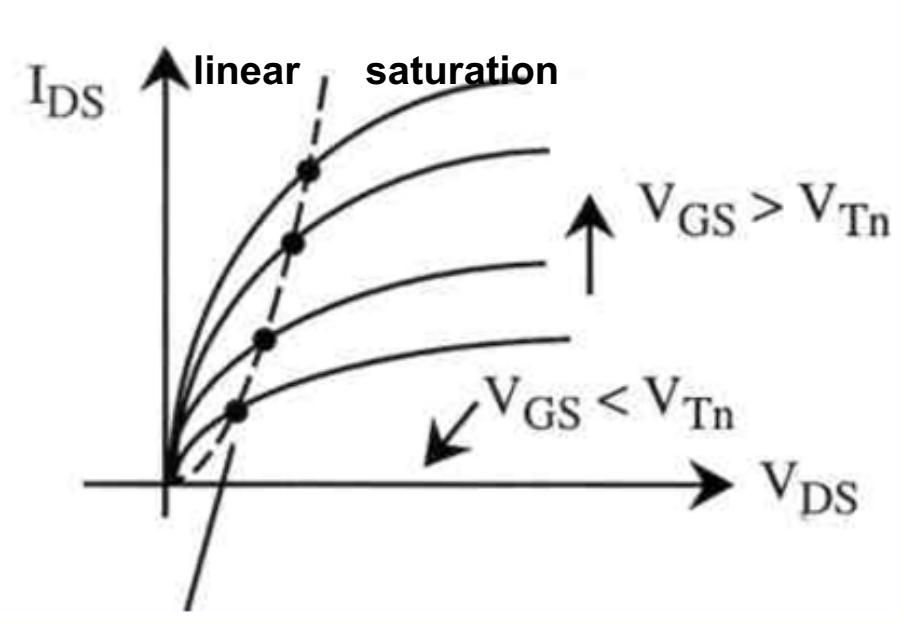
$$a_{33}^{(k)} = (a_{11}^{(k)} + a_{21}^{(k)} + a_{12}^{(k)} + a_{22}^{(k)})$$

PNP Bipolar Junction Transistor Model



MOSFET - NMOS





(i) *Subthreshold (also known as weak-inversion or cut-off) region:*

$V_{GS} < V_{th}$

$$I_D = I_{D0} e^{V_{ov}/\eta V_T}$$

$V_{ov} = V_{GS} - V_{th}$

V_{th} is the threshold voltage

V_{GS} is the gate to source voltage

$$V_{th} = V_{T0} + \gamma(\sqrt{V_{SB} + 2\phi} - \sqrt{2\phi})$$

(ii) *Linear region:*

$V_{GS} \geq V_{th}$ and $V_{DS} < V_{ov}$

$$I_D = k_n[V_{ov}V_{DS} - \frac{V_{DS}^2}{2}]$$

V_{DS} is the drain to source voltage

$k_n = \mu_n C_{ox} \frac{W}{L}$

μ_n is the charge-carrier effective mobility

W the gate width, L the gate length,

C_{ox} the gate oxide capacitance per unit area

(iii) *Saturation or active region:* $V_{GS} \geq V_{th}$ and $V_{DS} > V_{ov}$

$$I_D = \frac{1}{2} k_n V_{ov}^2 [1 + \lambda (V_{DS} - V_{ov})]$$

λ is a channel length modulation parameter.

when $V_{DS} = V_{ov}$

$$k_n [V_{ov} V_{DS} - \frac{V_{DS}^2}{2}] \quad = \quad \frac{1}{2} k_n V_{ov}^2 [1 + \lambda (V_{DS} - V_{ov})]$$

continuity of the equations across the regions

Taylor series expansion at iteration point $V_{GS}^{(k)}$ and $V_{DS}^{(k)}$

$$I = a_{11}^{(k)} V_{GS} + a_{12}^{(k)} V_{DS} + b_1^{(k)}$$

The coefficients are different in each region

$$\begin{bmatrix} 0 & 0 & 0 \\ +a_{11}^{(k)} & +a_{12}^{(k)} & -(a_{11}^{(k)} + a_{12}^{(k)}) \\ -a_{11}^{(k)} & -a_{12}^{(k)} & (a_{11}^{(k)} + a_{12}^{(k)}) \end{bmatrix} \begin{bmatrix} V_G \\ V_D \\ V_S \end{bmatrix}, \begin{bmatrix} 0 \\ -b_1^{(k)} \\ +b_1^{(k)} \end{bmatrix}$$

If $V_{SB} \neq 0$, V_{th} is evaluated at the beginning of every iteration.

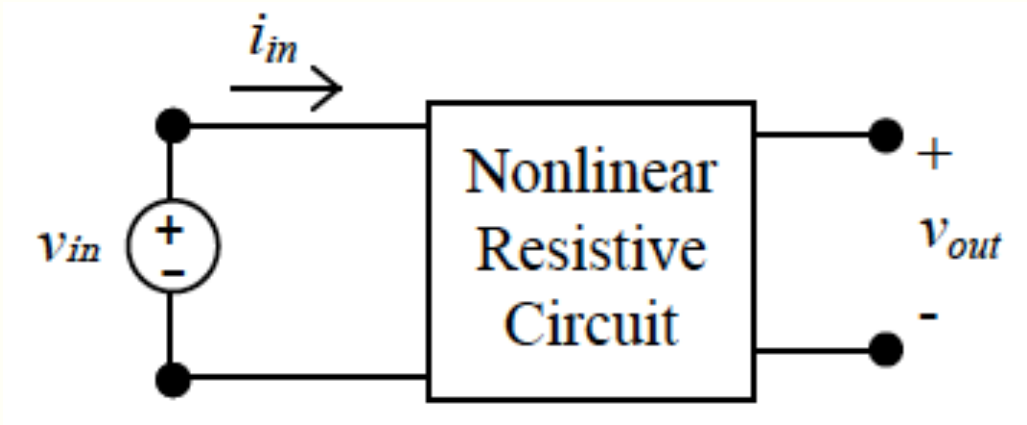
Taylor series expansion at $V_{GS}^{(k)}$, $V_{DS}^{(k)}$ and $V_{th}^{(k)}$

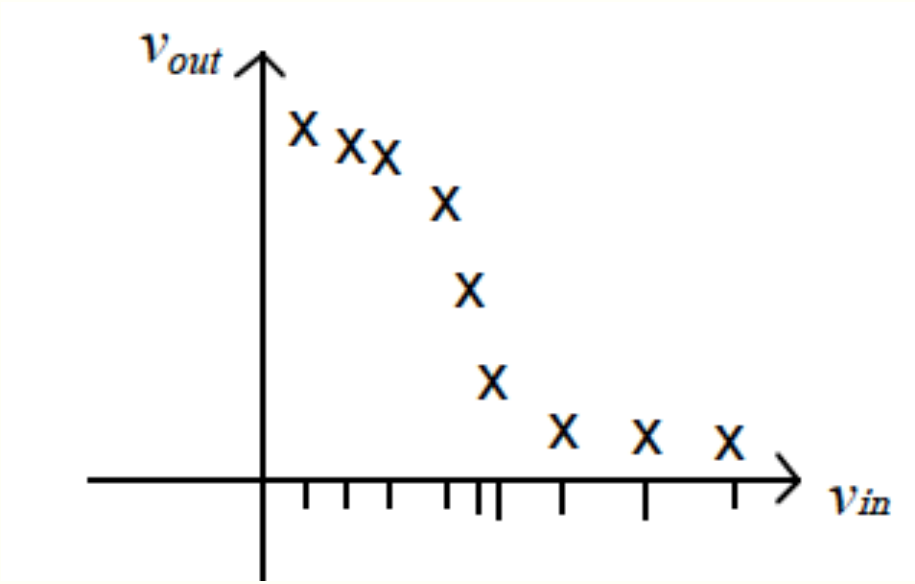
$$I_D = a_{11}^{(k)} V_{GS} + a_{12}^{(k)} V_{DS} + a_{13}^{(k)} V_{th} + b_1^{(k)}$$

$$V_{th} = a_{21}^{(k)} V_{SB} + b_2^{(k)}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \\ +a_{11}^{(k)} & +a_{12}^{(k)} & -(a_{11}^{(k)} + a_{12}^{(k)}) & +a_{13}^{(k)} & \\ -a_{11}^{(k)} & -a_{12}^{(\mathrm{k})} & (a_{11}^{(k)} + a_{12}^{(k)}) & -a_{13}^{(k)} & \\ 0 & 0 & -a_{21}^{(k)} & +1 & +a_{21}^{(k)} \end{bmatrix} \begin{bmatrix} V_G \\ V_D \\ V_S \\ V_{th} \\ V_B \end{bmatrix}, \begin{bmatrix} 0 \\ -b_1^{(k)} \\ +b_1^{(k)} \\ +b_2^{(k)} \end{bmatrix}$$

Input-Output DC Characteristics





Constructing $\mathbf{J}^{(k)} \triangle \mathbf{x}^{(k)} = \triangle \mathbf{y}^{(k)}$

Independent current source:

$$\triangle \mathbf{y}^{(k)}(n_1) \Leftarrow \triangle \mathbf{y}^{(k)}(n_1) - I$$

$$\triangle \mathbf{y}^{(k)}(n_2) \Leftarrow \triangle \mathbf{y}^{(k)}(n_2) + I$$

Independent voltage source:

$$\begin{array}{c} \begin{array}{cccccc} & n_1 & & n_2 & & \end{array} \\ \begin{bmatrix} \cdots & & \cdots & & \cdots & +1 \\ & \vdots & & \vdots & & \\ \cdots & & \cdots & & \cdots & -1 \\ & \vdots & & \vdots & & \\ & +1 & & -1 & & 0 \end{bmatrix} \end{array} \begin{bmatrix} \\ v_{n1} \\ \\ v_{n2} \\ \\ i_v \\ \\ \end{bmatrix}, \begin{bmatrix} rhs \\ \\ \\ \\ \\ E \\ \\ \end{bmatrix}$$

Independent voltage source:

$$\triangle \mathbf{y}^{(k)}(n_1) \Leftarrow \triangle \mathbf{y}^{(k)}(n_1) - i_v^{(k)}$$

$$\triangle \mathbf{y}^{(k)}(n_2) \Leftarrow \triangle \mathbf{y}^{(k)}(n_2) + i_v^{(k)}$$

$$\triangle \mathbf{y}^{(k)}(p) \Leftarrow \triangle \mathbf{y}^{(k)}(p) + E - (v_{n1}^{(k)} - v_{n2}^{(k)})$$

after the initial solution in the Newton method, $k \geq 1$, $E - (v_{n1}^{(k)} - v_{n2}^{(k)}) = 0$.

Voltage-controlled resistor:

$$i^{(k)} = f(v^{(k)})$$

$$\triangle \mathbf{y}^{(k)}(n_1) \Leftarrow \triangle \mathbf{y}^{(k)}(n_1) - i^{(k)}$$

$$\triangle \mathbf{y}^{(k)}(n_2) \Leftarrow \triangle \mathbf{y}^{(k)}(n_2) + i^{(k)}$$

Current-controlled resistor:

$$\begin{bmatrix} & n_1 & & n_2 & & \\ \cdots & & \cdots & & \cdots & +1 \\ & \vdots & & \vdots & & \\ \cdots & & \cdots & & \cdots & -1 \\ & \vdots & & \vdots & & \\ & +1 & & -1 & & -r \end{bmatrix} \begin{bmatrix} \\ v_{n1} \\ \\ v_{n2} \\ \\ i_r \\ \\ \end{bmatrix}, \begin{bmatrix} rhs \\ \\ \\ \\ \\ 0 \\ \\ \end{bmatrix}$$

$$\triangle \mathbf{y}^{(k)}(n_1) \Leftarrow \triangle \mathbf{y}^{(k)}(n_1) - i_r^{(k)}$$

$$\triangle \mathbf{y}^{(k)}(n_2) \Leftarrow \triangle \mathbf{y}^{(k)}(n_2) + i_r^{(k)}$$

$$\triangle \mathbf{y}^{(k)}(p) \Leftarrow \triangle \mathbf{y}^{(k)}(p) - (v_{n1}^{(k)} - v_{n2}^{(k)}) + v_r^{(k)}$$

Three-terminal voltage-controlled resistor:

Let $i_1^{(k)} = a_{11}v_1^{(k)} + a_{12}v_2^{(k)}$, $i_2^{(k)} = a_{21}v_1^{(k)} + a_{22}v_2^{(k)}$ in the linear case,

$i_1^{(k)} = f_1(v_1^{(k)}, v_2^{(k)})$ and $i_2^{(k)} = f_2(v_1^{(k)}, v_2^{(k)})$ in the nonlinear case

$$\triangle \mathbf{y}^{(k)}(n_1) \Leftarrow \triangle \mathbf{y}^{(k)}(n_1) - i_1^{(k)}$$

$$\triangle \mathbf{y}^{(k)}(n_2) \Leftarrow \triangle \mathbf{y}^{(k)}(n_2) - i_2^{(k)}$$

$$\triangle \mathbf{y}^{(k)}(n_3) \Leftarrow \triangle \mathbf{y}^{(k)}(n_3) + i_1^{(k)} + i_2^{(k)}$$

Partitioning and Relaxation Methods

Partitioning into Linear and Nonlinear Subcircuits

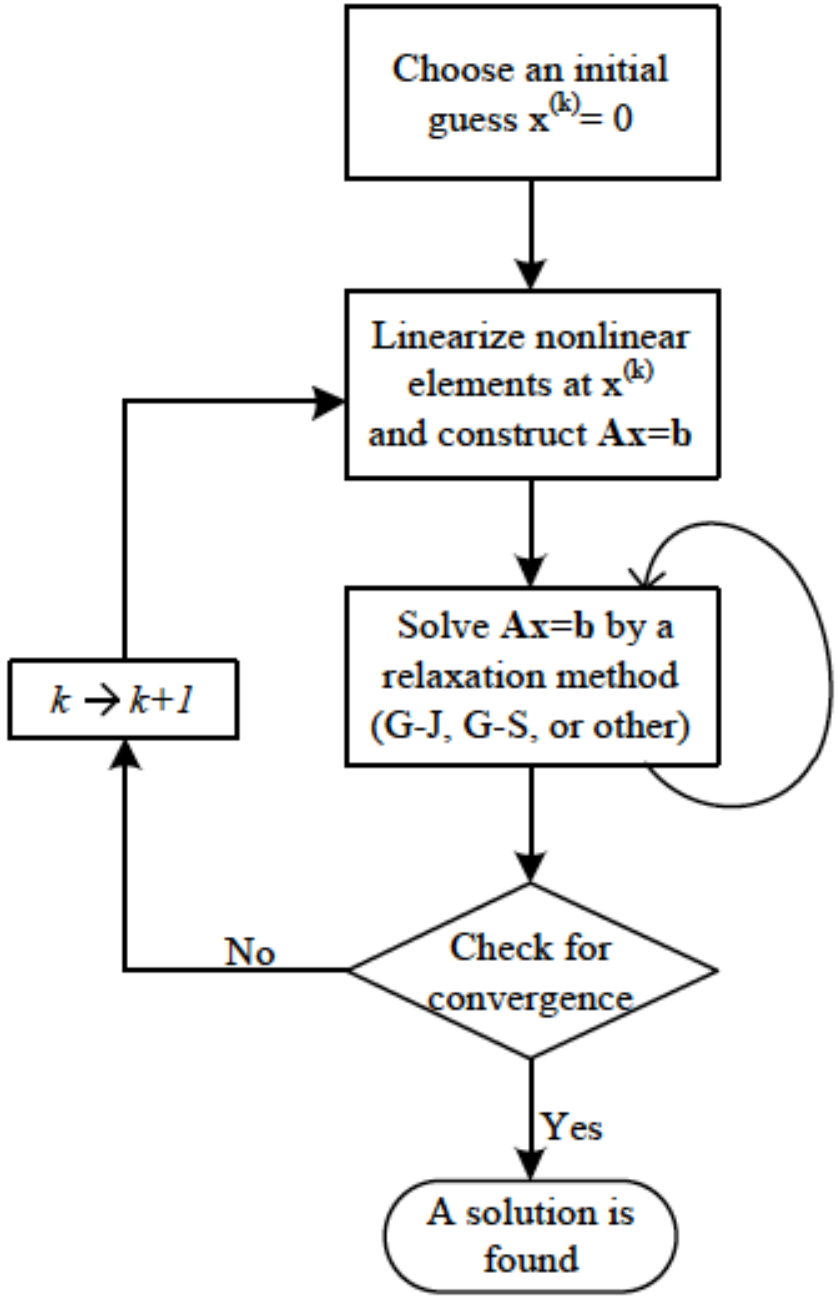
$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} & \mathbf{A}_{1\mathrm{t}} \\ \mathbf{0} & \mathbf{A}_{22}^{(k)} & \mathbf{A}_{2t}^{(k)} \\ \mathbf{A}_{\mathrm{t}1} & \mathbf{A}_{t2}^{(k)} & \mathbf{A}_{tt}^{(k)} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_t \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2^{(k)} \\ \mathbf{b}_t^{(k)} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{A}'_{1t} \\ \mathbf{0} & \mathbf{A}_{22}^{(k)} & \mathbf{A}_{2t}^{(k)} \\ \mathbf{0} & \mathbf{A}_{t2}^{(k)} & (\mathbf{A}_{tt}^{(k)} - \mathbf{H}_1) \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_t \end{bmatrix} = \begin{bmatrix} \mathbf{b}'_1 \\ \mathbf{b}_2^{(k)} \\ \mathbf{b}_t^{(k)} - \mathbf{b}'_1 \end{bmatrix}$$

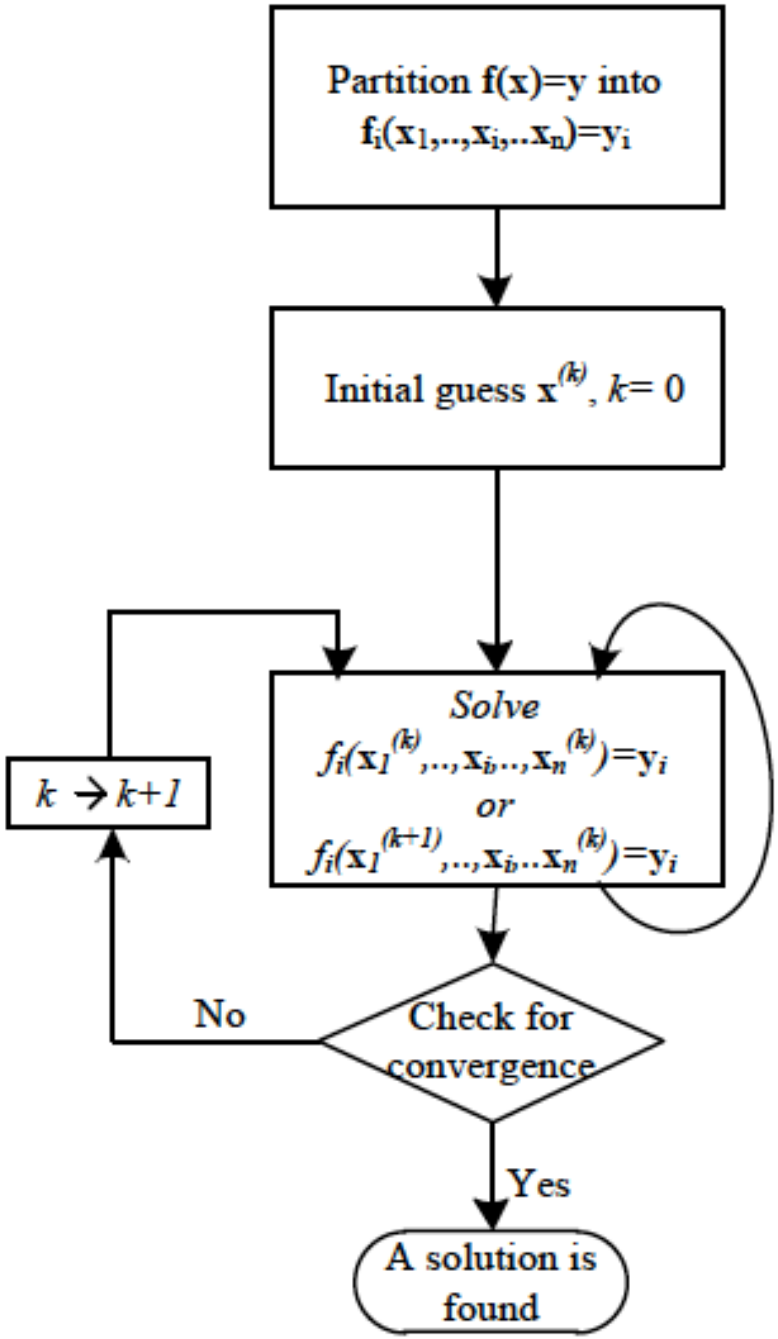
$$\begin{bmatrix} \mathbf{A}_{22}^{(k)} & \mathbf{A}_{2t}^{(k)} \\ \mathbf{A}_{t2}^{(k)} & (\mathbf{A}_{tt}^{(k)} - \mathbf{H}_1) \end{bmatrix} \begin{bmatrix} \mathbf{x}_2 \\ \mathbf{x}_t \end{bmatrix} = \begin{bmatrix} \mathbf{b}_2^{(k)} \\ \mathbf{b}_t^{(k)} - \mathbf{b}'_1 \end{bmatrix}$$

Relaxation Methods

Newton-Gauss-Jacobi and Newton-Gauss-Seidel



Gauss-Jacobi-Newton and Gauss-Seidel-Newton



Sensitivity Analysis

$$\mathbf{f}(\mathbf{x}, \mathbf{p}) = \mathbf{y}$$

$$\psi = \mathbf{e}^{\mathrm{T}}\mathbf{x}$$

$$\frac{\partial \psi}{\partial \mathbf{p}} = \mathbf{e}^{\mathrm{T}}\frac{\partial \mathbf{x}}{\partial \mathbf{p}}$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \mathbf{p}} + \frac{\partial \mathbf{f}}{\partial \mathbf{p}} = \frac{\partial \mathbf{y}}{\partial \mathbf{p}}$$

evaluated at $\mathbf{x} = \mathbf{x}^{(o)}$ and $\mathbf{p} = \mathbf{p}^{(o)}$

In most cases $\frac{\partial \mathbf{y}}{\partial \mathbf{p}} = \mathbf{0}$

$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \equiv \mathbf{J}$ has already been constructed and factorized into LU factors at $\mathbf{x} = \mathbf{x}^{(o)}$ and $\mathbf{p} = \mathbf{p}^{(o)}$

$$\frac{\partial \psi}{\partial \mathbf{p}} = \mathbf{e}^{\mathrm{T}} \frac{\partial \mathbf{x}}{\partial \mathbf{p}} = \mathbf{e}^{\mathrm{T}} \mathbf{J}^{-1} [\frac{\partial \mathbf{y}}{\partial \mathbf{p}} - \frac{\partial \mathbf{f}}{\partial \mathbf{p}}]$$

$$= \mathbf{u}^{\mathrm{T}} [\frac{\partial \mathbf{y}}{\partial \mathbf{p}} - \frac{\partial \mathbf{f}}{\partial \mathbf{p}}]$$

$\mathbf{u}^{\mathrm{T}} = \mathbf{e}^{\mathrm{T}} \mathbf{J}^{-1}$, or $[\mathbf{J}]^{\mathrm{T}} \mathbf{u} = \mathbf{e}$

If ψ is a nonlinear function of $\mathbf{x}$ and $\mathbf{p}$:

$$\psi = g(\mathbf{x}, \mathbf{p})$$

$$\frac{\partial \psi}{\partial \mathbf{p}} = \frac{\partial g}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{p}} + \frac{\partial g}{\partial \mathbf{p}} = \mathbf{e}_g^{\mathrm{T}} \frac{\partial \mathbf{x}}{\partial \mathbf{p}} + \frac{\partial g}{\partial \mathbf{p}}$$

$$\frac{\partial g}{\partial \mathbf{x}} \equiv \mathbf{e}_g^{\mathrm{T}}$$

Algorithm

1. Compute the solution of the resistive circuit at $\mathbf{p} = \mathbf{p}^{(o)}$

   {save the $\mathbf{LU}$ factors at the solution $\mathbf{x}^{(o)}$}

2. Construct $\mathbf{B} = \frac{\partial \mathbf{y}}{\partial \mathbf{p}} - \left[\frac{\partial \mathbf{f}}{\partial \mathbf{p}}\right]$ and $\frac{\partial g}{\partial \mathbf{x}} \equiv \mathbf{e}_g^{\mathrm{T}}$

3. Solve the transpose (or adjoint) system $\mathbf{U}^{\mathbf{T}}\mathbf{L}^{\mathbf{T}}\mathbf{u} = \mathbf{e}_g$

4. Find $\frac{\partial \psi}{\partial \mathbf{p}} = \mathbf{u}^{\mathbf{T}}\mathbf{B} + \frac{\partial g}{\partial \mathbf{p}}$